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UNIFORMLY CONVERGENT STOLZ THEOREM FOR SEQUENCES OF FUNCTIONS

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Abstract

Based on the lemma from Toplitz transformation for sequences of uniformly convergent functions, we derive and prove the uniformly convergent Stolz theorem for sequences of functions in region *I*.

1. Introduction

In this paper, we start from Stolz theorem for infinite sequences and generalize it to Stolz theorem for sequences of functions. First we give the following theorems for

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infinite sequences:

Theorem 1 (Stolz Theorem [1-5])**.** *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences such that*

(1) $\{a_n\}$ *is a strictly monotone increasing sequence*;

(2) $\lim\limits_{n \to \infty} a_n = +\infty$.

If $\lim\limits_{n \to \infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l$ *exists* (*having a limit of positive and negative infinity*), *then* $\lim\limits_{n \to \infty} \frac{b_n}{a_n} = l$.

There is no requirement for sequence $\{b_n\}$ to have a limit of ∞, so this theorem could be remarked as $\frac{*}{\infty}$ Stolz theorem. Similarly, we generalize Stolz theorem for sequences of uniformly convergent functions with indeterminate expression as following:

Theorem 2. *If two sequences of functions* $\{f_n(x)\}$ *and* $\{g_n(x)\}$ *satisfy the following conditions in region I*,

(1) $g_n(x)$ *is uniformly convergent to* $+\infty$ on region *I*;

(2) *For any* $x \in I$, $0 < \{g_n(x)\}$ *monotone increases*;

(3) *For any given n*, *function* $\frac{f_{n+1}(x) - f_n(x)}{g_{n+1}(x) - g_n(x)}$. *is bounded on region I*.

Then, *if* $\frac{f_{n+1}(x) - f_n(x)}{g_{n+1}(x) - g_n(x)}$ *is uniformly convergent to* $h(x)$, $\frac{f_n(x)}{g_n(x)}$ *is uniformly convergent to* $h(x)$.

Similar to the proof of Stolz theorem for unlimited sequences, we need to give Toplitz transformation for sequences of functions which are the following three lemmas.

Lemma 1. *Let sequences of functions* $\{f_n(x)\}$, $\{g_n(x)\}$, *and* $\{a_{ij}(x)\}$ *be* *defined in region I, if for any* $x \in I$, *we have*:

$$\begin{pmatrix} g_1(x) \\ g_2(x) \\ \dots \\ g_n(x) \\ \dots \end{pmatrix} = \begin{pmatrix} a_{11}(x) & 0 & \cdots & 0 & \cdots \\ a_{21}(x) & a_{22}(x) & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ \cdots \\ f_n(x) \\ \cdots \end{pmatrix},$$

and the following hold;

(1) $a_{ij}(x) \geq 0,\ j < i,\ i,\ j = 1, 2, L, \forall x \in I$;

(2) $\sum_{j=1}^{i} a_{ij}(x) = 1, \forall i \in N, \forall x \in I$;

(3) *For any* $j \in N$, $x \in I$; a_{nj} *is uniformly convergent to* 0 *when n approaches infinity*;

(4) *For any* n, $f_n(x)$ *is bounded in region I.*

Then, if $\{f_n(x)\}$ *is uniformly convergent to* $h(x)$, $x \in I$, *we have*

$$\sum_{j=1}^{n} a_{nj}(x) f_j(x) = g_n(x) \Rightarrow h(x), \quad n \to \infty,\ x \in I.$$

Proof. For any n, $f_n(x)$ is bounded in region I, if $\{f_n(x)\}$ is uniformly convergent to $h(x)$, then $\{f_n(x)\}$ and $h(x)$ are uniformly bounded in region I, that means there exists $M > 0$ such that for any n, $x \in I$, $|f_n(x)| \leq M$. Further because $\{f_n(x) - h(x)\}$ is uniformly convergent to 0, then for $\varepsilon = 1$, there exists $N > 0$, for any $n > N$, $x \in I$, $\{f_n(x) - h(x)\} < 1$.

Let $M = \max\{M_1, M_2, \dots, M_N, 1\}$, where M_i is the bound for $f_i - h(x)$ in region I, thus for any $x \in I$, n, we have

$$|f_n(x) - h(x)| \le M.$$

As $\{f_n(x)\}$ is uniformly convergent to $h(x)$, we know that for any $\varepsilon > 0$, there exists a N_0, for any $n > N_0$, $x \in I$, we have

$$|f_n(x) - h(x)| < \frac{\varepsilon}{2}.$$

Moreover, for any n, $\{a_{nj}(x)\}$ is uniformly convergent to 0, we have $\left\{M \sum_{j=1}^{N_0} a_{nj}(x)\right\}$ uniformly convergent to 0, therefore for the above $\frac{\varepsilon}{2}$, there exists a N_1, for any $n > N_1$, $x \in I$,

$$M \sum_{j=1}^{N_0} a_{nj}(x) < \frac{\varepsilon}{2}.$$

Let $N = N_0 + N_1$, when $n > N$, for any $x \in I$, we have:

$$|g_n(x) - h(x)| = \left|\sum_{j=1}^{n} a_{nj}(x) f_j(x) - h(x)\right| = \left|\sum_{j=1}^{n} a_{nj}(x) f_j(x) - h(x) \sum_{j=1}^{n} a_{nj}(x)\right|$$

$$= \left|\sum_{j=1}^{n} a_{nj}(x)\big(f_j(x) - h(x)\big)\right| \le \sum_{j=1}^{n} |a_{nj}(x)| \cdot |f_j(x) - h(x)|$$

$$= \sum_{j=1}^{N_0} |a_{nj}(x)| \cdot |f_j(x) - h(x)| + \sum_{j=N_0+1}^{n} |a_{nj}(x)| \cdot |f_j(x) - h(x)|$$

$$\le M \sum_{j=1}^{N_0} |a_{nj}(x)| + \frac{\varepsilon}{2} \sum_{j=N_0+1}^{n} |a_{nj}(x)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So we have

$$\sum_{j=1}^{n} a_{nj}(x) f_j(x) = g_n(x) \Rightarrow h(x),\ n \to \infty, \quad x \in I.$$

Thus Lemma 1 holds.

Lemma 2. *Let sequences of functions* $\{f_n(x)\}$, $\{g_n(x)\}$, *and* $\{a_{ij}(x)\}$ *defined in region I, for any* $x \in I$, $g_n(x) = \sum_{j=1}^{n} a_{nj}(x) f_j(x)$ *exists and following hold*;

(1) *There exists* $k > 0$, *for any* $n \in N$, $x \in I$, $|a_{n1}(x)| + |a_{n2}(x)| + \ldots + |a_{nn}(x)| \leq k$;

(2) *For any* $j \in N$, $x \in I$, $a_{nj}(x)$ *is uniformly convergent to* 0, $n \to \infty$;

(3) $A_n(x) = a_{n1}(x) + a_{n2}(x) + \ldots + a_{nn}(x)$ *is uniformly convergent to* 1 *as* $n \to \infty$, $x \in I$;

(4) *For any* n, $f_n(x)$ *is bounded in region I.*

Then, if $\{f_n(x)\}$ *is uniformly convergent to* 0, $n \to \infty$, $x \in I$, *we have*

$$\sum_{j=1}^{n} a_{nj}(x) f_j(x) = g_n(x) \Rightarrow 0, \quad n \to \infty, \ x \in I.$$

Proof. Since $f_n(x)$ is uniformly convergent to 0, and for any n, $f_n(x)$ is bounded in region I, we know that $\{f_n(x)\}$ is uniformly bounded in region I, which means that there exists $M > 0$, for any $j \in N$, $\{a_{nj}(x)\}$ uniformly convergent to 0, and we have $|f_n(x)| \leq M$. Similarly, for any $\varepsilon > 0$, there exists a N_0, for any $n > N_0$, $x \in I$, we have $|f_n(x)| < \frac{\varepsilon}{2k}$. Since for any n, $|a_{nj}(x)|$ is uniformly convergent to 0, further we know $\left\{ M \sum_{j=1}^{n} |a_{nj}(x)| \right\}$ is uniformly convergent to 0, thus for the above $\varepsilon > 0$, there exists a $N_1 > 0$, for any $n > N_1$, $x \in I$ we have

$$M \sum_{j=1}^{n} |a_{nj}(x)| \leq \frac{\varepsilon}{2}.$$

Let $N = N_0 + N_1$, when $n > N$, for any $x \in I$, we have

$$|g_n(x)| = \left|\sum_{j=1}^{n} a_{nj}(x) f_j(x)\right| \leq \sum_{j=1}^{N_0} |a_{nj}(x)| \cdot |f_j(x)| + \sum_{j=N_0+1}^{N} |a_{nj}(x)| \cdot |f_j(x)|$$

$$\leq M \sum_{j=1}^{N_0} |a_{nj}(x)| + \frac{\varepsilon}{2k} \sum_{j=N_0+1}^{N} |a_{nj}(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2k} \cdot k = \varepsilon.$$

Thus Lemma 2 holds.

Lemma 3. *Let sequences of functions* $\{f_n(x)\}$, $\{g_n(x)\}$, *and* $\{a_{ij}(x)\}$ *be defined in region I. If for any* $x \in I$, *we have* $g_n(x) = \sum_{j=1}^{n} a_{nj}(x) f_j(x)$ *and following hold*:

(1) *There exists* $k > 0$, *for any* $n \in N$, $x \in I$, *we have* $|a_{n1}(x)| + |a_{n2}(x)| + \ldots + |a_{nn}(x)| \leq k$;

(2) *For any* $j \in N$, $x \in I$, $\{a_{nj}(x)\}$ *is uniformly convergent to* 0, $n \to \infty$;

(3) $A_n(x) = a_{n1}(x) + a_{n2}(x) + \ldots + a_{nn}(x)$ *is uniformly convergent to* 1 *as* $n \to \infty$, $x \in I$;

(4) *For any* n, $f_n(x)$ *is bounded in region I.*

Then, if $\{f_n(x)\}$ *is uniformly convergent to* $h(x)$, $x \in I$, *we have*

$$\sum_{j=1}^{n} a_{nj}(x) f_j(x) = g_n(x) \Rightarrow h(x), \quad n \to \infty, \ x \in I.$$

Proof. Let $A_n(x) = 1 + \alpha_n(x)$, then $\{\alpha_n(x)\}$ is uniformly convergent to 0, we can get

$$g_n(x) - h(x) = \sum_{j=1}^{n} a_{nj}(x) f_j(x) [A_n(x) - \alpha_n(x)] - h(x)$$

$$= a_{n1}(x)(f_1(x) - h(x)) + a_{n2}(x)(f_2(x) - h(x))$$

$$+\ldots + a_{nn}(x)(f_n(x) - h(x)) + h(x)\alpha_n(x)$$

$$= Z_n(x) + h(x)\alpha_n(x).$$

Since $\{f_n(x)\}$ is uniformly convergent to $h(x)$, we can conclude that $\{f_n(x) - h(x)\}$ is uniformly convergent to 0, hence $\{Z_n(x)\}$ is uniformly convergent to 0, then

$$g_n(x) - h(x) = Z_n(x) + h(x)\alpha_n(x).$$

As for any n, $f_n(x)$ is bounded in region I, we know that $h(x)$ is bounded, which means there exists a $M > 0$ such that for any $x \in I$, $|h(x)| \leq M$. On the other hand, $\alpha_n(x)$ is uniformly convergent to 0, so, for any $\varepsilon > 0$, there exists a $N > 0$ such that for any $n > N$, $x \in I$, we have $|\alpha_n(x)| < \varepsilon$,

$$|h(x)\alpha_n(x)| \leq M|\alpha_n(x)| < M\varepsilon.$$

Thus $\{h(x)\alpha_n(x)\}$ is uniformly convergent to 0, furthermore, we can conclude that $\{g_n(x) - h(x)\}$ is uniformly convergent to 0, and $\{g_n(x)\}$ is uniformly convergent to $h(x)$. Thus Lemma 3 holds.

Now we use the above lemmas to prove Theorem 2.

Proof. Let $z_n(x) = \dfrac{f_{n+1}(x) - f_n(x)}{g_{n+1}(x) - g_n(x)}$ be uniformly convergent to $h(x)$ and $a_{nj}(x) = \dfrac{g_j(x) - g_{j-1}(x)}{g_n(x)}$, then

$$w_n(x) = a_{n1}(x)z_1(x) + a_{n2}(x)z_2(x) + \ldots + a_{nn}(x)z_n(x) = \frac{f_n(x)}{g_n(x)} - \frac{f_0(x)}{g_n(x)}.$$

Since $\{g_n(x)\}$ is uniformly convergent to $+\infty$, so we have $a_{nj}(x) = \dfrac{g_j(x) - g_{j-1}(x)}{g_n(x)}$ is uniformly convergent to 0 $(n \to \infty,\ x \in I)$, then

$$A_n(x) = a_{n1}(x) + a_{n2}(x) + \ldots + a_{nn}(x) = \frac{g_n(x) - g_0(x)}{g_n(x)}$$

is uniformly convergent to 1. Then from above Lemma 3, we have

$$w_n(x) = \sum_{j=1}^{n} a_{nj}(x) z_j(x) \Rightarrow h(x), \quad n \to \infty,\ x \in I.$$

Moreover, since $\left\{\frac{f_0(x)}{g_n(x)}\right\}$ is uniformly convergent to 0, we have that $\left\{\frac{f_n(x)}{g_n(x)}\right\}$ is uniformly convergent to $h(x),\ n \to \infty,\ x \in I.$ Finally, Theorem 2 holds.

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