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TAIL-DEPENDENCE PROPERTIES OF SOME NEW TYPES OF COPULA MODELS (PART II)

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Abstract

We continue the investigation of the tail-dependence behaviour of some new types of copula models, published recently in [5] and [6].

1. Introduction

For the sake of simplicity, we concentrate our investigations to the two-dimensional case. Let $U_1,\ U_2$ be standard random variables, i.e., they follow a uniform distribution over the interval $[0, 1]$ each. Let further $T_1,\ T_2$ be real continuous functions over $\mathbb{R}^2$ and $W_1 = T_1(U_1, U_2),\ W_2 = T_2(U_1, U_2)$. If $W_1,\ W_2$ already follow a continuous uniform distribution over $[0, 1]$ each, then (W_1, W_2) is a representative of

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a two-dimensional copula. Otherwise, $(V_1, V_2) := (F_1(W_1), F_2(W_2))$ is a representative of a two-dimensional copula if F_i denotes the continuous c.d.f. of W_i, $i = 1, 2$.

Of particular interest especially for financial markets or risk management is the tail dependence of copulas which was explicitly treated for dependence-of-unity copulas in [2], [3] and [4], and for the new approach in [6], which we shall continue here. The simplest definition of the coefficient λ_U of upper and λ_L of lower tail dependence is

$$\lambda_U = \lim_{t \uparrow 1} \frac{P(W_1 > F_1^{-1}(t),\, W_2 > F_2^{-1}(t))}{1-t},$$

$$\lambda_L = \lim_{t \downarrow 0} \frac{P(W_1 \le F^{-1}(t),\, W_2 \le G^{-1}(t))}{t},$$

see, e.g., [1], Def. 7.36, p. 247.

In case that $F_1 = F_2$, i.e., W_1 and W_2 have same distribution, we also have

$$\lambda_U = \lim_{s \uparrow \infty} \frac{P(W_1 > s,\, W_2 > s)}{1 - F(s)}, \qquad \lambda_L = \lim_{s \downarrow -\infty} \frac{P(W_1 \le s,\, W_2 \le s)}{F(s)}.$$

2. Particular Cases

Case 1. Here we consider the choice

$$W_1 = T_1(U_1, U_2) = U_1 + U_2, \qquad W_2 = T_2(U_1, U_2) = \max(U_1, U_2).$$

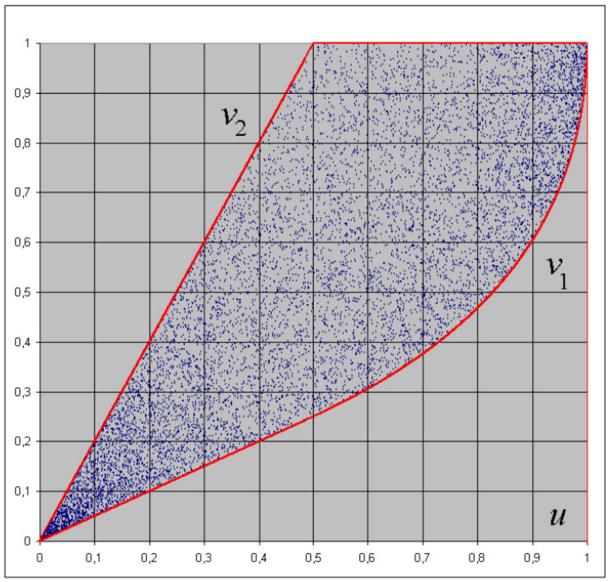
It is easy to see that the corresponding c.d.f.'s are given by

$$F_1(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \le x \le 1, \\ 1 - 2\left(1 - \dfrac{x}{2}\right)^2, & 1 \le x \le 2 \end{cases}$$

and

$$F_2(x) = x^2, \quad 0 \leq x \leq 1.$$

The following graph shows 10,000 simulations of $(V_1, V_2) = (F_1(W_1), F_2(W_2))$.



The red lines (u, v) represent the (sharp) lower and upper envelopes of the copula, which are given by

$$v_1 = v_{lower} = \begin{cases} \dfrac{u}{2}, & \text{if } u \leq \dfrac{1}{2}, \\ \left(1 - \sqrt{\dfrac{1-u}{2}}\right)^2, & \text{otherwise} \end{cases}$$

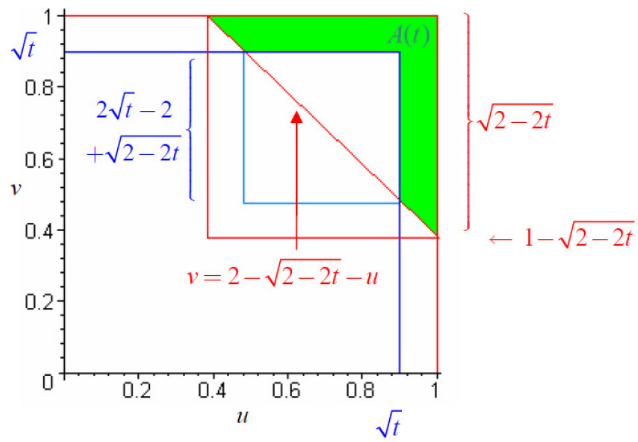
and

$$v_2 = v_{upper} = \begin{cases} 2u, & \text{if } u \leq \dfrac{1}{2}, \\ 1, & \text{if } u > \dfrac{1}{2}. \end{cases}$$

The lower bound is reached if V_1 and V_2 are close to each other, while the upper bound is reached if one of V_1 or V_2 is close to zero.

The subsequent graph explains our arguments for the calculation of the coefficient λ_U of upper tail dependence, which is given by $\lambda_U = 0$.

We start with some preliminary inequalities.



We have, for $t > \frac{1}{2}$, with μ denoting Lebesgue measure,

$$P(W_1 > F_1^{-1}(t), W_2 > F_2^{-1}(t))$$

$$= P(\max(U_1, U_2) > \sqrt{t}, U_2 > 2 - \sqrt{2 - t} - U_1)$$

$$= \mu(A(t)) = \frac{1}{2}(\sqrt{2 - 2t}^2 - (2\sqrt{t} - 2 + \sqrt{2 - 2t})^2)$$

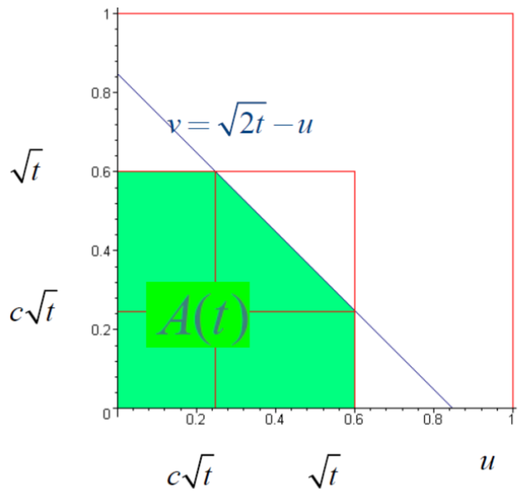
or, by a Taylor expansion around the point $t = 1$,

$$\mu(A(t)) = \sqrt{2}(1 - t)^{3/2} + \mathcal{O}((1 - t)^2), \text{ hence } \lambda_U = \lim_{t \to 1} \frac{\mu(A(t))}{1 - t} = 0,$$

as stated.

The subsequent graph explains our arguments for the calculation of the coefficient λ_L of lower tail dependence, which is given by $\lambda_L = 2(\sqrt{2}-1) = 0.828427\ldots$.

We start again with some preliminary inequalities.



We have, for $t > \frac{1}{2}$, with $c = \sqrt{2} - 1$,

$$P(\max(U_1, U_2) \leq F_2^{-1}(t), U_1 + U_2 \leq F_1^{-1}(t))$$

$$= P(\max(U_1, U_2) \leq \sqrt{t}, U_2 \leq \sqrt{2t} - U_1)$$

$$= \mu(A(t)) = t - \frac{\{(1-c)\sqrt{t}\}^2}{2} = t\left(1 - \frac{(1-c)^2}{2}\right) = 2ct$$

and hence $\lambda_L = \lim_{t \downarrow 0} \frac{\mu(A(t))}{t} = 2c = 2(\sqrt{2}-1) = 0.828427\ldots$, as stated.

Case 2. Here we consider the choice

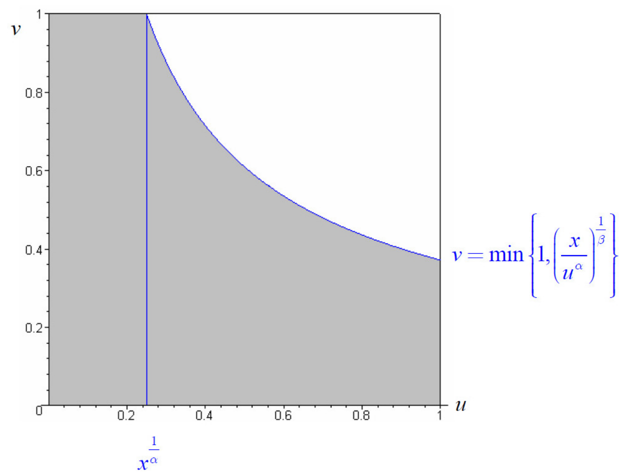
$W_1 = T_1(U_1, U_2) = U_1^{\alpha} U_2^{\beta}$, $W_2 = T_2(U_1, U_2) = U_1^{\beta} U_2^{\alpha}$ with real $\alpha, \beta > 0$.

For simplicity, let $U := U_1$, $V := U_2$.

The common c.d.f. of U and V is given by

$$F(x) = \frac{\alpha}{\alpha-\beta}x^{\frac{1}{\alpha}} - \frac{\beta}{\alpha-\beta}x^{\frac{1}{\beta}}, \qquad 0 < x < 1$$

which can be seen as follows.



For $0 < x < 1$, there holds

$$F(x) = P(U^\alpha V^\beta \le x) = P\left(V \le \left(\frac{x}{U^\alpha}\right)^{\frac{1}{\beta}}\right)$$

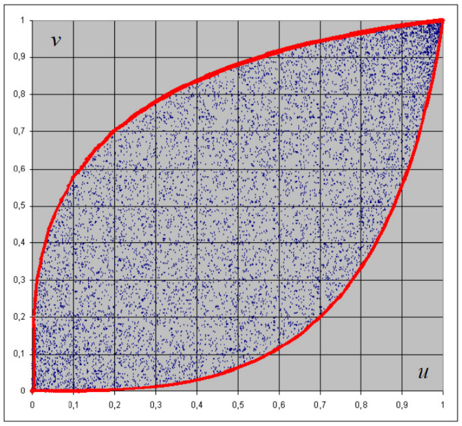
$$= \int_0^1 P\left(V \le \left(\frac{x}{u^\alpha}\right)^{\frac{1}{\beta}}\right)du = \int_0^1 \min\left\{1, \left(\frac{x}{u^\alpha}\right)^{\frac{1}{\beta}}\right\}du$$

$$= x^{\frac{1}{\alpha}} + x^{\frac{1}{\beta}} \int_{x^{\frac{1}{\alpha}}}^{1} \frac{1}{u^{\frac{\alpha}{\beta}}}du = x^{\frac{1}{\alpha}} + \frac{\beta}{\beta-\alpha}x^{\frac{1}{\beta}}\left[1 - x^{\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)}\right]$$

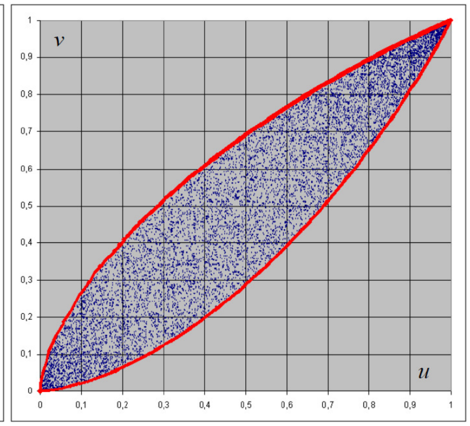
$$= \frac{\alpha}{\alpha-\beta}x^{\frac{1}{\alpha}} - \frac{\beta}{\alpha-\beta}x^{\frac{1}{\beta}} = P(U^\beta V^\alpha \le x)$$

by symmetry reasons.

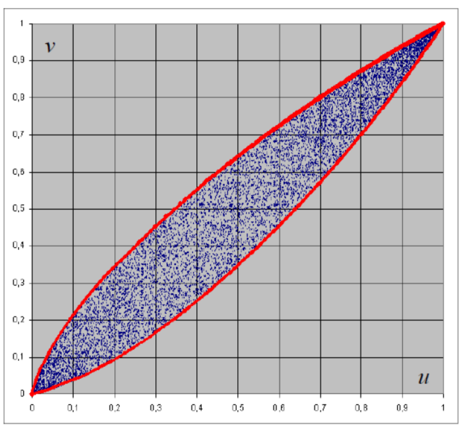
The following graphs show $10{,}000$ simulations each of the copula given by $(F(W_1), F(W_2))$, for different values of α and β.



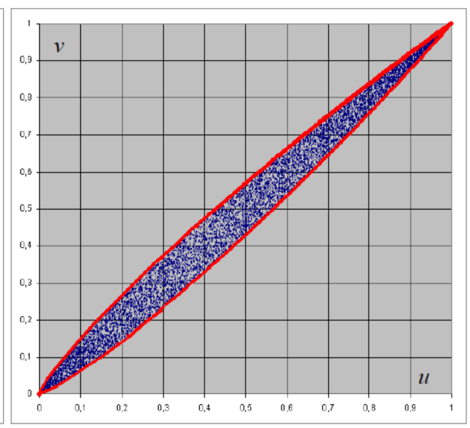
$\alpha = 3, \beta = 1$



$\alpha = 3, \beta = 2$



$\alpha = 4, \beta = 3$



$\alpha = 4, \beta = 3.5$

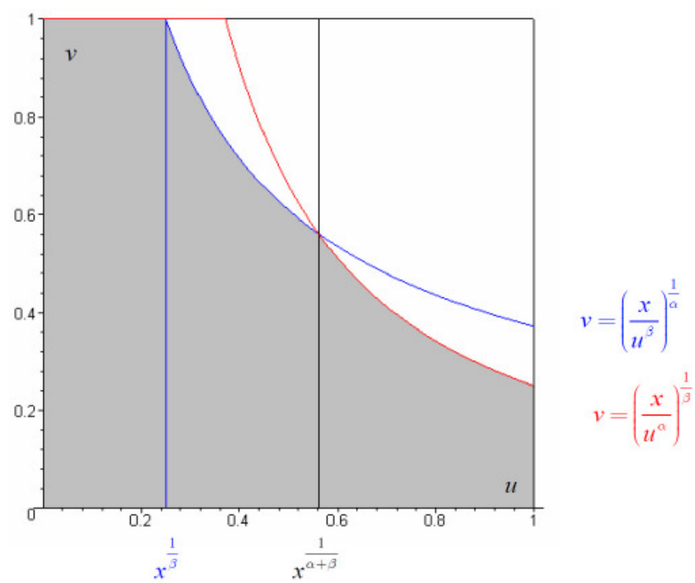
The red lines (u, v) represent the (sharp) lower and upper envelopes of the copula, which are given by

$$v_{lower} = F\left(\left(F^{-1}(u)\right)^{\frac{\beta}{\alpha}}\right) \quad \text{and} \quad v_{upper} = F\left(\left(F^{-1}(u)\right)^{\frac{\alpha}{\beta}}\right), \qquad 0 < u < 1.$$

Alternatively, the lower envelope can be described by the points $\left[F(u),\ F\left(u^{\frac{\beta}{\alpha}}\right)\right]$ and the upper envelope by the points $\left[F(u),\ F\left(u^{\frac{\alpha}{\beta}}\right)\right],$ $0 < u < 1.$

Note that for $\alpha >> \beta,$ the copula tends to the independence copula, and for $\alpha \approx \beta,$ we obtain the upper Fréchet bound. Note also that the copula is symmetric in $\alpha,\ \beta.$

The subsequent graph explains our arguments for the calculation of the coefficient λ_L of lower tail dependence, which is given by $\lambda_L = 0.$ First notice that for $0 < x < 1,$ the intersection point of $\left(\frac{x}{u^{\alpha}}\right)^{\frac{1}{\beta}}$ and $\left(\frac{x}{u^{\beta}}\right)^{\frac{1}{\alpha}}$ is given by $u_x = x^{\frac{1}{\alpha+\beta}}$ with value $u_x.$



Next we have

$$P\left(U^{\alpha}V^{\beta} \leq x,\, U^{\beta}V^{\alpha} \leq x\right)$$

$$= x^{\frac{1}{\beta}} + \int_{x^{\frac{1}{\beta}}}^{x^{\frac{1}{\alpha+\beta}}} \left(\frac{x}{u^{\beta}}\right)^{\frac{1}{\alpha}} du + \int_{x^{\frac{1}{\alpha+\beta}}}^{1} \left(\frac{x}{u^{\alpha}}\right)^{\frac{1}{\beta}} du = x^{\frac{2}{\alpha+\beta}}$$

with

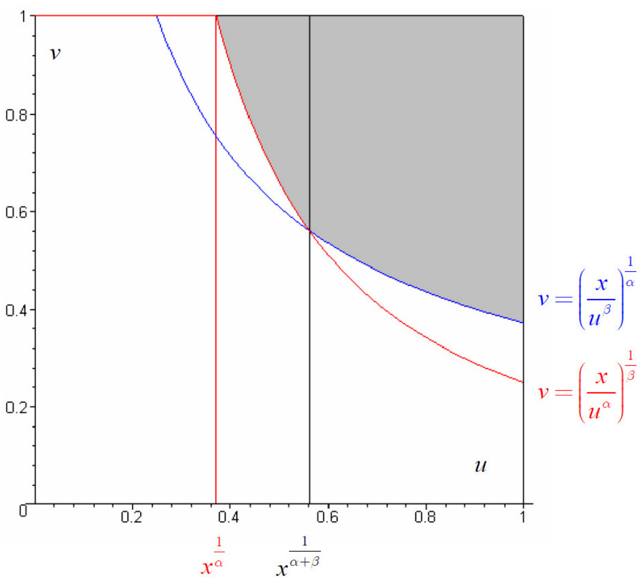
$$\frac{F(x)}{P\left(U^{\alpha}V^{\beta} \leq x,\, U^{\beta}V^{\alpha} \leq x\right)} = \frac{\alpha}{\alpha-\beta}\, x^{\frac{\beta-\alpha}{\alpha\cdot(\alpha+\beta)}} - \frac{\beta}{\alpha-\beta}\, x^{\frac{\alpha-\beta}{\alpha\cdot(\alpha+\beta)}}$$

and hence

$$\lim_{x\to 0} \frac{F(x)}{P\left(U^{\alpha}V^{\beta} \leq x,\, U^{\beta}V^{\alpha} \leq x\right)} = \infty, \text{ i.e.,}$$

$$\lim_{x\to 0} \frac{P\left(U^{\alpha}V^{\beta} \leq x,\, U^{\beta}V^{\alpha} \leq x\right)}{F(x)} = 0 = \lambda_L.$$

The subsequent graph explains our arguments for the calculation of the coefficient λ_U of upper tail dependence, which is given by $\lambda_U = \frac{2\beta}{\alpha+\beta} > 0$ if $\alpha > \beta$.



If $\alpha > \beta$,

$$P\left(U^\alpha V^\beta > x,\, U^\beta V^\alpha > x\right) = 1 - x^{\frac{1}{\alpha}} - \int_{x^{\frac{1}{\alpha}}}^{x^{\frac{1}{\alpha+\beta}}} \left(\frac{x}{u^\alpha}\right)^{\frac{1}{\beta}} du - \int_{x^{\frac{1}{\alpha+\beta}}}^{1} \left(\frac{x}{u^\beta}\right)^{\frac{1}{\alpha}} du$$

$$= 1 - \frac{2\alpha}{\alpha-\beta} x^{\frac{1}{\alpha}} + \frac{\beta}{\alpha-\beta} x^{\frac{2}{\alpha+\beta}}$$

with, by a Taylor expansion around $x = 1$,

$$\frac{P\left(U^\alpha V^\beta > x,\, U^\beta V^\alpha > x\right)}{1 - F(x)} = \frac{2\beta}{\alpha+\beta} - \frac{2(\alpha-\beta)}{3(\alpha+\beta)^2}(x-1) + \mathcal{O}\left((x-1)^2\right),$$

hence $\lambda_U = \dfrac{2\beta}{\alpha+\beta} > 0.$

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