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SIMPLIFICATION OF AUTO-BACKLUND TRANSFORMATION OF SINE-GORDON EQUATION AND ITS NEW EXACT SOLUTION

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Abstract

Sine-Gordon (SG) equation is a second order nonlinear partial differential equation (NPDE) proposed by Backlund in 1876 as a model for nonlinear pseudo-spherical surface and its exact solution found by reducing to two coupled first order NPDEs and method called Auto-Backlund Transformation (ABT). In this study, above ABT is further simplified to a pair of coupled nonlinear first order ordinary differential equations by the method of Lie Group Similarity Transformation and found same exact solution that Backlund reported as well as a new exact solution of SG equation is also reported.

Keywords and phrases: Sine-Gordon, Lie group, similarity transformation, integrability, Auto-Backlund transformation, inverse scattering transformation, Painleve property, pseudo spherical surfaces.

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1. Introduction

The contact Transformation of Differential Geometry [1] provides Lie Group Similarity Transformation in nonlinear differential equations [2] by which one can construct various classes of similar equations [2, 3]. Backlund reported [4, 5] two simultaneous first order differential equations arising in differential geometry so that by contact Transformation one can find class of similar equations corresponding to a given nonlinear partial differential equation (NPDE), thereby integrals of original equation can be found out. This method is called Auto-Backlund Transformation (ABT) if two integrals are of same equations are connected [5, 6, 7].

Sine-Gordon (SG) equation [4, 5] proposed by Backlund is a two dimensional $(1+1)$ can be written as

$$u_{xt} = \sin(u) \tag{1.01}$$

as a model for nonlinear pseudo-spherical surface of constant negative curvature. Same equation studied by Lamb [8, 9] as a model of propagation of ultra short optical pulses. Painleve Property (PP) [10] Inverse Scattering Transformation (IST), Lax Pairs [11] are known and so considered as a completely integrable system. A well known exact solution of SG equation reported by Backlund is

$$u(x, t) = 4 \arctan[\exp(kx - wt)]. \tag{1.02}$$

Backlund found above solution by converting (1.01) into a pair of simultaneous first order NPDEs

$$v_x + u_x = (2/c)\sin[(u - v)/2] \tag{1.03}$$

and

$$u_t - v_t = 2c \sin[(u + v)/2], \tag{1.04}$$

where u and v are two distinct exact solutions of SG equation (1.01) and c is an arbitrary constant.

This study reports simplification of (1.03) and (1.04) into a pair of simultaneous first order nonlinear ordinary differential equation by the method of Lie Group Similarity Transformation [2] and reproduces the same exact solution (1.02) in more general form, also found new exact solution of SG equation (1.01).

2. Lie Group Similarity Transformation Method for Partial Differential Equation

Essential details of the Lie continuous point group similarity transformation method to reduce the number of independent variables of a partial differential equation (PDE) so as to obtain respective ordinary differential equation (ODE) [6] is the following. Let the given PDE in two independent variables x and t and one dependent variable u be

$$F(x, t, u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.1}$$

where $u_t, u_x, \ldots$ are all partial derivatives of dependent variables $u(x, t)$ with respect to the independent variable t and x, respectively.

When we apply a family of one parameter infinitesimal continuous point group transformations,

$$x = x + \varepsilon X(x, t, u) + O(\varepsilon^2), \tag{2.2}$$

$$t = t + \varepsilon T(x, t, u) + O(\varepsilon^2), \tag{2.3}$$

$$u = u + \varepsilon U(x, t, u) + O(\varepsilon^2), \tag{2.4}$$

we get the infinitesimals of the variables u, t and x as U, T, X, respectively and ε is an infinitesimal parameter. The derivatives of u are also transformed as

$$u_x = u_x + \varepsilon[U_x] + O(\varepsilon^2), \tag{2.5}$$

$$u_{xx} = u_{xx} + \varepsilon[U_{xx}] + O(\varepsilon^2), \tag{2.6}$$

$$u_{tt} = u_{tt} + \varepsilon[U_{tt}] + O(\varepsilon^2), \tag{2.7}$$

where $[U_x]$, $[U_{xx}]$, $[U_{tt}]$ are the infinitesimals of the derivatives u_x, u_{xx}, u_{tt}, respectively. These are called first and second extensions and are given by [2]

$$[U_x] = U_x + (U_u - X_x)u_x - X_u u_x^2 - T_x u_t - T_x u_x u_t, \tag{2.8}$$

$$\begin{aligned}[U_{xx}] = U_{xx} &+ (2U_{xu} - X_{xx})u_x + (U_{uu} - 2X_{xu})u_x^2 - X_{uu}u_x^3 \\ &+ U_u - 2X_x u_{xx} - 3X_u u_x u_{xx} - T_{xx}u_t - 2T_{xu}u_x u_t - T_{uu}u_x^2 u_t \\ &-2T_x u_{xt} - T_u u_{xx} u_t - 2T_u u_{xt} u_x,\end{aligned} \tag{2.9}$$

$$\begin{aligned}[U_{tt}] = U_{tt} &+ [2U_{tu} - T_{tt}]u_t - X_{tt}u_x + [U_{uu} - 2T_{uu}]u_t^2 \\ &- 2X_{tu}u_x u_t - T_{uu}u_t^3 - X_{uu}u_t^2 u_x + [U_u - 2T_t]u_{tt} - 2X_t u_{xt} \\ &-3T_u u_{tt} u_t - X_u u_{tt} u_x - 3X_u u_{xt} u_t.\end{aligned} \tag{2.10}$$

The invariant requirements of given PDE (2.1) under the set of above transformations lead to the invariant surface conditions,

$$\begin{aligned}T\frac{\partial F}{\partial t} + X\frac{\partial F}{\partial x} + U\frac{\partial F}{\partial u} + [U_x]\frac{\partial F}{\partial u_x} \\ + [U_{tt}]\frac{\partial F}{\partial u_{tt}} + [U_{xx}]\frac{\partial F}{\partial u_{xx}} = 0.\end{aligned} \tag{2.11}$$

On solving above invariant surface condition (2.11), the infinitesimals X, T, U can be uniquely obtained, that give the similarity group under which the given PDE (2.1) is invariant. This gives

$$T\frac{du}{dt} + X\frac{du}{dx} - \frac{du}{dU} = 0. \tag{2.12}$$

The solution of (2.12) are obtained by Langrange's condition,

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U}. \tag{2.13}$$

This yields,

$$x = x(t, C_1, C_2)$$

and

$$u = u(t, C_1, C_2), \tag{2.14}$$

where C_1 and C_2 are arbitrary integration constants and the constant C_1 plays the role of an independent variable called the similarity variable S and C_2 that of a dependent variable called the similarity solution $u(S)$ such that exact solution of given PDE, so that

$$u(x, t) = u(S). \tag{2.15}$$

On substituting (2.15) in given PDE (2.1) reduces to an ordinary differential equation with S as independent variable and $u(S)$ as dependent variable.

3. Simplification of ABT by Lie Group Similarity Transformation

The general form of the simultaneous equations (1.03) and (1.04) is given by

$$F(t, x, u, v, v_x, u_x, u_t, v_t) = 0. \tag{3.01}$$

The invariant surface conditions for (3.01) is

$$T\frac{\partial F}{\partial t}+X\frac{\partial F}{\partial x}+U\frac{\partial F}{\partial u}+V\frac{\partial F}{\partial v}+[U_x]\frac{\partial F}{\partial u_x}$$

$$+[U_t]\frac{\partial F}{\partial u_t}+[V_x]\frac{\partial F}{\partial v_x}+[V_t]\frac{\partial F}{\partial v_t}=0, \qquad (3.02)$$

where $[U_x]$, $[U_t]$, $[V_x]$, $[V_t]$ are the first extensions of the partial derivatives of $[u_x]$, $[u_t]$, $[v_x]$, $[v_t]$ for two dependent variables [2] that are the following

$$[U_x]=U_x+U_uu_x+U_vv_x-(T_xu_t+X_xu_x)$$

$$-(T_uu_xu_t+T_vv_xu_t)-(X_uu_xu_x+X_vv_xu_x), \qquad (3.03)$$

$$[V_x]=V_x+V_uu_x+V_vv_x-(T_xv_t+X_xv_x)$$

$$-(T_uu_xv_t+T_vv_xv_t)-(X_uu_xv_x+X_vv_xv_x), \qquad (3.04)$$

$$[U_t]=U_t+U_uu_t+U_vv_t-(X_tu_x+T_tu_t)$$

$$-(T_uu_tu_t+T_vv_tu_t)-(X_uu_tu_x+X_vu_xv_t), \qquad (3.05)$$

$$[V_t]=V_t+V_uu_t+V_vv_t-(X_tv_x+T_tv_t)$$

$$-(T_uu_tv_t+T_vv_tu_t)-(X_uu_tu_x+X_vu_xv_t). \qquad (3.06)$$

Substitute (3.03), (3.04), (3.05), (3.06) in the invariant surface condition (3.02) and collect same orders of derivatives of u, and v, then we get the following constraints for $U = V = 0$,

$$T_u=0,\ T_t=0,\ T_v=0,\ T_x=0, \qquad (3.07)$$

$$X_u=0,\ X_v=0,\ X_x=0,\ X_t=0. \qquad (3.08)$$

On solving above constrained equations, we get

$$X=w, \qquad (3.09)$$

$$T = k, \tag{3.10}$$

where w and k are arbitrary integration constants. Then substitute (3.09) and (3.10) in the Lagrange's conditions (2.13)

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U} = \frac{dv}{V}, \tag{3.11}$$

then we get the similarity variable $z(x, t)$ as

$$z(x, t) = (kx - wt). \tag{3.12}$$

Then the similarity solution of the simultaneous differential equations (1.03) and (1.04) are

$$u(x, t) = u(z), \tag{3.13}$$

and

$$v(x, t) = v(z). \tag{3.14}$$

On substituting (3.13) and (3.14) in the ABT of SG equation (1.03) and (1.04), we get the following coupled simultaneous first order ordinary differential equations

$$\frac{du}{dz} + \frac{dv}{dz} = \left(\frac{2}{ck}\right)\sin\left[\frac{(u - v)}{2}\right], \tag{3.15}$$

and

$$\frac{dv}{dz} - \frac{du}{dz} = \left(\frac{2c}{w}\right)\sin\left[\frac{(u + v)}{2}\right]. \tag{3.16}$$

Since $u(x, t) = 0$ is an exact solution of SG equation (1.01), substituting $u = 0$ in (3.15) and (3.16) we can find an exact solution of SG equation (1.01) as

$$v(x, t) = -\pi + 4 \arctan[kx - wt + C], \tag{3.17}$$

where $k = 2/c$ and $w = 2c$ and C is an integration constant. Above exact solution of SG is more general form than the known solution (1.02).

4. Discussion

Auto-Backlund transformation of SG equation (1.01) also connect four exact solutions without any derivatives, but trigonometric relationships [1, 3] as

$$\tan\left[\frac{(u_3 - u_0)}{4}\right] = \left[\frac{(a+b)}{(c+d)}\right]\tan\left[\frac{(u_1 + u_2)}{4}\right], \tag{4.01}$$

where u_0, u_1, u_2, and u_3 are any four exact solutions of SG equation (1.01) and a, b, c, and d are arbitrary constants. So that when u_0, u_1, and u_2 are known then the fourth solution u_3 can be found out from (4.01). $u(x, t) = 0$ is a solution of SG equation, so let $u_0 = 0$, also from (1.02) we have the well-known solution (1.02), so $u_1 = 4\arctan[\exp(kx - wt)]$ be the second solution.

Recently this author reported [12] a new solution of SG equation using Lie group similarity transformation in which similarity variable $s(x, t)$

$$s(x, t) = [-c^2xt + c(kx - vt) + kv]. \tag{4.02}$$

Then the exact solution of SG equation (1.01) is

$$u(x, t) = u(s), \tag{4.03}$$

where $u(s)$ is

$$u(s) = 4\arctan\left(4\sqrt{s}\right). \tag{4.04}$$

Then the third solution u_2 of SG equation is (4.03) and the fourth new solution can be obtained from the trigonometric relation (4.01) as

$$\tan\left(\frac{u_3}{4}\right) = \left[\frac{(a+b)}{(c+d)}\right]\tan\left[\arctan[\exp(kx - vt)] - 4\arctan\left(\sqrt{s}\right)\right]. \quad (4.05)$$

This process of finding new solutions of SG equation (1.01) can be continued for any number of times, but complexity of solutions is also increasing more and more as above.

References

[1] W. F. Ames, Nonlinear Partial Differential Equations in Engineering, Vol. 2, Academic Press, New York, 1972.

[2] G. M. Bluman and J. D. Cole, Similarity Methods for Differential Equations, Springer-Verlag, Berlin, 1974.

[3] L. P. Eisenhart, Treatise on the Differential Geometry of Curves and Surfaces, Dover, New York, 1960.

[4] A. V. Backlund, Math. Ann. 17 (1880), 285.

[5] A. V. Backlund, Math. Ann. 9 (1876), 297.

[6] M. J. Clarin, Ann. Sci. Ecole Norm. Sup. 3e Ser. Suppl. 19 (1902), s1.

[7] M. J. Clarin Ann. Fac. Sci. Univ. Toulouse 2e Ser. 5 (1903), 437.

[8] G. L. Lamb, Jr., Phys. Lett. 25A (1967), 181.

[9] G. L. Lamb, Jr., Phys. Lett. 28A (1969), 548.

[10] John Weiss, M. Tabor and George Carnevale, J. Math. Phys. 24 (1983), 522.

[11] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonliear Evolution Equations and Inverse Scattering, Cambridge Univ. Press, 1991.

[12] B. V. Baby, Fundamental J. Math. Phys. 10(1) (2022), 25-33.