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QUASI AL-WATANI MANIFOLDS

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Abstract

Quasi AL-Watani manifolds have been defined, and some of their geometric properties are derived. Also a non-trivial example of quasi AL-Watani manifold has been introduced to prove its existence.

1. Introduction

Chaki [1] introduced the notion of a quasi Einstein manifold, whose Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition:

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{1.1}$$

where a and b are scalars of which $b \neq 0,$ and A is a non-zero 1-form such that

$$g(X, \rho) = A(X), \tag{1.2}$$

for all vector fields X, ρ being a unit vector field.

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In a recent paper [2], the author introduced a special type of Riemannian manifold called AL-Watani manifold, i.e., a Riemannian manifold $(M^n, g)(n \geq 2)$, such that its curvature tensor R satisfies the relation:

$$R(X, Y, Z) = a[S(Y, Z)X + g(Y, Z)QX], \tag{1.3}$$

where a is constant and Q is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S such that

$$S(X, Y) = g(QX, Y). \tag{1.4}$$

The object of this paper is to study a special type of Riemannian manifold $(M^n, g)(n \geq 2)$ such that its curvature tensor R satisfies:

$$R(X, Y, Z) = a[S(Y, Z)X + g(Y, Z)QX] + bA(Y)A(Z)X, \tag{1.5}$$

where $a, b,$ and A are as stated above. Such a manifold shall be called quasi AL-Watani manifold. In particular, if the 1-form A vanishes identically then the quasi AL-Watani manifold will then reduce to AL-Watani manifold. This will justify the definition and the name quasi AL-Watani manifold.

It is known [3] that a Riemannian manifold is of Codazzi type Ricci tensor and of cyclic Ricci tensor if the following relations, respectively, hold:

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z), \tag{1.6}$$

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = 0. \tag{1.7}$$

In Section 2, it is shown that every quasi AL-Watani manifold is a quasi Einstein manifold and on quasi AL-Watani manifold if the Ricci tensor is of Codazzi type then the 1-form A is closed iff the scalar curvature is constant, whereas if the Ricci tensor is of cyclic type then the integral curves of the vector field ρ are gradient. Also it is shown that if the vector field ρ is a killing vector field on quasi AL-Watani manifold then the Ricci tensor is cyclic type if and only if the scalar curvature is constant. Last section is devoted to a non trivial example of quasi AL-Watani manifold to prove the existence.

2. Quasi AL-Watani Manifolds

Contracting (1.5) with respect to X we get

$$S(Y, Z) = -\frac{bn}{(an-1)} A(Y)A(Z) - \frac{ar}{(an-1)} g(Y, Z). \tag{2.1}$$

This shows that every quasi AL-Watani manifold is a quasi Einstein manifold. Thus we can state

Theorem 2.1. *Every quasi AL-Watani manifold is quasi Einstein manifold.*

Taking covariant derivative of the above equation we get

$$(\nabla_X S)(Y, Z) = -\frac{bn}{(an-1)}[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)]$$

$$-\frac{adr(X)}{(an-1)} g(Y, Z). \tag{2.2}$$

If the manifold is of Codazzi type Ricci tensor then we have from (2.2) and (1.6),

$$\frac{bn}{(an-1)}[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)] + \frac{adr(X)}{(an-1)} g(Y, Z)$$

$$= \frac{bn}{(an-1)}[(\nabla_Y A)(X)A(Z) + (\nabla_Y A)(Z)A(X)] + \frac{adr(Y)}{(an-1)} g(X, Z). \tag{2.3}$$

Let $Z = \rho$ in (2.3) and taking in account $(\nabla_X A)(\rho) = 0,$ we can have

$$\frac{bn}{(an-1)}[(\nabla_X A)(Y) - (\nabla_Y A)(X)]A(\rho)$$

$$= \frac{a}{(an-1)}[dr(X)A(Y) - dr(Y)A(X)]. \tag{2.4}$$

If we consider r to be constant then since $A(\rho) \neq 0,$ (2.4) will reduce to

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0. \tag{2.5}$$

This means that the 1-form A is closed. Conversely if the 1-form A is closed then (2.4) will reduce to

$$dr(X)A(Y) - dr(Y)A(X) = 0. \tag{2.6}$$

This gives

$$dr(X) = 0.$$

Thus we can state

Theorem 2.2. *On quasi AL-Watani manifold of Codazzi type Ricci tensor the* 1-*form A is closed if and only if it is of constant scalar curvature*.

From (1.7) and (2.2) we have

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y)$$

$$= -\frac{bn}{(an-1)}[((\nabla_X A)(Y) + (\nabla_Y A)(X))A(Z) + ((\nabla_X A)(Z)$$

$$+(\nabla_Z A)(X))A(Y) + ((\nabla_Z A)(Y) + (\nabla_Y A)(Z))A(X)]$$

$$-\frac{a}{(an-1)}[dr(X)g(Y, Z) + dr(Y)g(X, Z) + dr(Z)g(X, Y)]. \tag{2.7}$$

If we consider quasi AL-Watani manifold to be of cyclic Ricci tensor and of constant scalar curvature then contracting (2.7) with respect to Y and Z, we can get

$$(\nabla_\rho A)(X) = 0, \tag{2.8}$$

which implies that $\nabla_\rho \rho = 0.$ Thus by hypothesis we can state

Theorem 2.3. *On quasi AL-Watani manifold of constant scalar curvature if the Ricci tensor is of cyclic type then the integral curves of the vector field* ρ *are gradient*.

Now let us consider the generator vector field ρ to be a killing vector field of the quasi AL-Watani manifold. Then we have

$$(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0. \tag{2.9}$$

Using this equation on (2.7), we get

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y)$$

$$= -\frac{a}{(an-1)}[dr(X)g(Y, Z) + dr(Y)g(X, Z) + dr(Z)g(X, Y)]. \tag{2.10}$$

Thus we can state

Theorem 2.4. *On quasi AL-Watani manifold if the vector field* ρ *is a killing vector field then the Ricci tensor is cyclic type if and only if the scalar curvature is constant*.

3. Example of Quasi AL-Watani Manifold

Let us consider R^4 endowed with the Riemannian metric [4]

$$d^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2, \qquad (3.1)$$

where $i, j = 1, 2, 3, 4.$ Then it is known [4] that the only non vanishing Ricci tensors and the curvature tensors are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}; \quad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = \frac{-2}{3(x^4)^{\frac{1}{3}}},$$

$$R_{1441} = R_{2442} = R_{3443} = \frac{-2}{9(x^4)^{\frac{2}{3}}}, \qquad (3.2)$$

$$S_{11} = S_{22} = S_{33} = \frac{-2}{9(x^4)^{\frac{2}{3}}}; \quad S_{44} = \frac{-2}{3(x^4)^2}, \qquad (3.3)$$

and the scalar curvature $r = \dfrac{-4}{3(x^4)^2}$.

Let us define A_i, a and b as follows:

$$A_i = \frac{-3}{(x^4)^2}, \text{ for } i = 1, 2, 3, 4 \qquad (3.4)$$

$$a = \frac{1}{8}; \quad b = \frac{1}{3}. \qquad (3.5)$$

To verify the definition by (1.5), we have to verify only the following relations:

$$R_{1441} = a[S_{44}g_{11} + S_{11}g_{44}] + bA_4A_4g_{11}, \qquad (3.6)$$

$$R_{2442} = a[S_{44}g_{22} + S_{22}g_{44}] + bA_4A_4g_{22}, \qquad (3.7)$$

$$R_{3443} = a[S_{44}g_{33} + S_{33}g_{44}] + bA_4A_4g_{33}. \tag{3.8}$$

Using (3.2), (3.3), (3.4) and (3.5) on (3.6), we get

$$\text{R.H.S.} = a[S_{44}g_{11} + S_{11}g_{44}] + bA_4A_4g_{11}$$

$$= \frac{1}{8}\left[\frac{-2}{3(x^4)^2}(x^4)^{\frac{4}{3}} + \frac{-2}{9(x^4)^{\frac{2}{3}}}(1)\right] + \frac{1}{3}\left[\frac{-3}{(x^4)^2}(x^4)^{\frac{4}{3}}\right]$$

$$= \frac{-2}{9(x^4)^{\frac{2}{3}}} = \text{L.H.S.}$$

Similarly, we can show (3.6) and (3.7) are true, whereas the other cases are trivially true. Hence R^4 along with the metric g defined by (3.1) is quasi AL-Watani manifold. Thus we can state

Theorem 3.1. *A Riemannian manifold* (M^4, g) *endowed with the metric* (3.1) *is a quasi AL-Watani manifold with non-constant scalar curvature*.

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