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PARTITIONS OF WIGNER $3\text{-}j$ OR SUPER $3\text{-}j^S$ SYMBOLS INDUCED BY REGGE SYMMETRY: ACCURATE DESCRIPTION

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Abstract

We show that Regge transformations induce five partitions on $\mathfrak{su}(2)$ $(3\text{-}j)$ symbols, namely $S_p(0)$, $S_p(1)$, $S_p(2)$, $S_p(4)$, $S_p(5)$, with an empty set $S_p(3) = \varnothing$. The super-algebra $\mathfrak{osp}(1|2)$ admits three kinds of super-symbols $(3\text{-}j)^S_\alpha$, $(3\text{-}j)^S_\beta$, $(3\text{-}j)^S_\gamma$, whose α, β, γ parities are fixed by the values of $2(j \pm m)$. We find also five partitions for $\mathfrak{osp}(1|2)$ $(3\text{-}j)^S_\alpha$, $(3\text{-}j)^S_\gamma$ symbols, but they reduce to $S_p(0)$, $S_p(1)$ for a $(3\text{-}j)^S_\beta$. Unexpectedly, this latter and its 'Regge-transformed' may be opposite in sign. A formula fully similar to that of a $(3\text{-}j)$ is derived for the $(3\text{-}j)^S$. Some *forbidden* $(3\text{-}j)^S_\beta$ require an analytic prolongation, consistent with Regge β-partitions, which enables to establish a complete $(3\text{-}j)^S$ table.

Keywords and phrases: angular momentum in quantum mechanics, Wigner $3j$-symbol, partition, Regge symmetry, supersymmetry, Racah-Wigner calculus, $S3\text{-}j$, super $3\text{-}j^S$ symbol.

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1. Introduction

Since the historical discovery (1958) of Regge symmetries for Clebsch-Gordan coefficients [1], we only learned that the initial symmetry group $\overset{\#=12}{S_p}$ of Wigner $3\text{-}j$ symbols becomes a larger group of order 72. All this was analyzed in Table of Rotenberg et al. [2], in standard books like [3] or in special topics about angular momentum [4], for example. This is well known, but what is not known at all is what are the quantitative parameters that determine all intermediate groups of order 12 up to 72. Indeed the linear transformations of Regge induce partitions on the $3\text{-}j$ symbols via these parameters. This is the subject of the present paper.

It should be noted that the idea of partitions related to Regge transformations is mentioned only one time at the end of a paper by Shelepin in 1964 [5], but no research in this sense appears to have been pursued to date, after checking the (rare) most recent articles on the $3\text{-}j$ symbols.

As in our recent work on $\{6\text{-}j\}$ symbols partitions [6], the aim is to carry out a similar analysis on $(3\text{-}j)$ symbols. The first task is to find the right partition parameters for the $(3\text{-}j)$ symbols.

A priori, they are far to be apparent data. However, it well seems that they are involved in the analytic formulas themselves [6], here under the form of $(j_k \pm m_k)$, where k refers to the kth column, $k = [1, 3]$. This key-parameter allows one to define a 'column-parity' even or odd according to the parity of $2(j_k \pm m_k)$, respectively. Any column $c_k = \begin{vmatrix} j_k \\ m_k \end{vmatrix}$ can be of two kinds, denoted by a shorthand notation like $\overset{|ev|}{c_k}$ or $\overset{|od|}{c_k}$. Alternative notations are possible:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left(\begin{vmatrix} j_1 \\ m_1 \end{vmatrix} \begin{vmatrix} j_2 \\ m_2 \end{vmatrix} \begin{vmatrix} j_3 \\ m_3 \end{vmatrix} \right) = (c_1 \quad c_2 \quad c_3). \tag{1}$$

For $\mathfrak{su}(2)$ any $(3\text{-}j)$ is of kind $(\overset{|ev|}{c_1}\ \overset{|ev|}{c_2}\ \overset{|ev|}{c_3})$. This will be different for $o\mathfrak{sp}(1|2)$ and

$(3\text{-}j)^S$ symbols. As will be seen further the concept of 'column-parity' naturally leads to properly assign intrinsic parities to super $(3\text{-}j)^S$ symbols [7, 8] and classify their Regge-partitions. The paper is organized as follows: Sections 2-3 are devoted to $(3\text{-}j)^S$ symbols, Sections 4-5 to super $(3\text{-}j)^S$ symbols and Section 6 to an analytic prolongation of some 'forbidden' super symbols.

2. Analytic Formula for $(3\text{-}j)$ Symbols

For Wigner $3\text{-}j$ symbols, denoted here by $(3\text{-}j)$, the most commonly used expression $[2, 3, 9, 10]$ can be written down as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\Bigg|_{\sum_k m_k = 0} = \Delta(j_1 j_2 j_3) v \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\Bigg|_{\sum_k m_k = 0}, \quad (2.1)$$

where Δ triangle of Edmonds [9, p. 99] has been used here for convenience

$$\Delta(abc) = \left(\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}\right)^{1/2}, \quad (2.2)$$

and v is directly arranged with $(j_k \pm m_k)$ parameters announced in introduction:

$$v\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+m_1-(j_2-m_2)}\left(\prod\nolimits_k (j_k + m_k)!(j_k - m_k)!\right)^{\frac{1}{2}}$$

$$\times \sum_z \frac{(-1)^z}{z!(z-(j_2+m_2-(j_3-m_3)))!(z-((j_1-m_1)-(j_3+m_3)))! \times} . \quad (2.3)$$
$$\times (j_1 + m_1 + j_2 + m_2 - (j_3 - m_3)) - z)!(j_1 - m_1 - z)!(j_2 + m_2 - z)!$$

This is nothing more than that used in Ref. [2][1] for computing $(3\text{-}j)$ symbols numerical values.

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[1]Misprints: in (1.11) no frontal phase, in rhs of (1.12), m_3 to be replaced by m_2.

3. Regge Symmetry of $(3\text{-}j)$ Symbols

According to our analysis done with $\{6\text{-}j\}$ symbols [6], we are interested in the production of new triangles $(j_1' j_2' j_3')$ from a given $(j_1 j_2 j_3)$. We shall write out in detail only the relevant transformations by avoiding phase factors in formulas, which is possible using one of the twelve $(3\text{-}j)$ symmetries. Our notations of the partition parameters will be the following

$$j_k^+ = (j_k + m_k), \quad j_k^- = (j_k - m_k). \tag{3.1}$$

As a matter of fact, a glance at the Regge array [2, 3], also used to represent a $(3\text{-}j)$ symbol, directly shows the underlying existence of these parameters:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = R = \begin{bmatrix} -j_1 + j_2 + j_3 & j_1 - j_2 + j_3 & j_1 + j_2 - j_3 \\ j_1 - m_1 & j_2 - m_2 & j_3 - m_3 \\ j_1 + m_1 & j_2 + m_2 & j_3 + m_3 \end{bmatrix}, \tag{3.2}$$

$$R = \begin{bmatrix} R_1^1 & R_1^2 & R_1^3 \\ R_2^1 & R_2^2 & R_2^3 \\ R_3^1 & R_3^2 & R_3^3 \end{bmatrix} = \begin{bmatrix} -j_1^- + j_2^+ + j_3^+ & j_1^+ - j_2^- + j_3^+ & j_1^+ + j_2^+ - j_3^- \\ j_1^- & j_2^- & j_3^- \\ j_1^+ & j_2^+ & j_3^+ \end{bmatrix}. \tag{3.3}$$

Below *five* Regge transformations are listed from $\mathcal{R}_1$ up to $\mathcal{R}_5$. They generate at most 5 distinct triangles different from the original. This means also 5 distinct $(3\text{-}j)$ symbols, of course with the same numerical value. We emphasize this point because for super $(3\text{-}j)^S$ symbols it may happen that numerical values do not have the same sign.

Overview of Regge transformations

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & \frac{1}{2}(j_3^- + j_2^-) & \frac{1}{2}(j_3^+ + j_2^+) \\ (j_2 - j_3) & \frac{1}{2}(j_3^- - j_2^-) & \frac{1}{2}(j_3^+ + j_2^+) \end{pmatrix}, \qquad \mathcal{R}_1 \tag{3.4}$$

$$= \begin{pmatrix} \frac{1}{2}(j_1^- + j_3^-) & j_2 & \frac{1}{2}(j_1^+ + j_3^+) \\ \frac{1}{2}(j_1^- - j_3^-) & (j_3 - j_1) & \frac{1}{2}(j_1^+ - j_3^+) \end{pmatrix}, \qquad \mathcal{R}_2 \; (3.5)$$

$$= \begin{pmatrix} \frac{1}{2}(j_2^- + j_1^-) & \frac{1}{2}(j_2^+ + j_1^+) & j_3 \\ \frac{1}{2}(j_2^- - j_1^-) & \frac{1}{2}(j_2^+ - j_1^+) & (j_1 - j_2) \end{pmatrix}. \qquad \mathcal{R}_3 \; (3.6)$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(j_3^- + j_2^-) & \frac{1}{2}(j_1^- + j_3^-) & \frac{1}{2}(j_2^- + j_1^-) \\ \frac{1}{2}(j_3^- + j_2^-) - j_1^+ & \frac{1}{2}(j_1^- + j_3^-) - j_2^+ & \frac{1}{2}(j_2^- + j_1^-) - j_3^+ \end{pmatrix}, \qquad \mathcal{R}_4 \; (3.7)$$

$$= \begin{pmatrix} \frac{1}{2}(j_3^+ + j_2^+) & \frac{1}{2}(j_1^+ + j_3^+) & \frac{1}{2}(j_2^+ + j_1^+) \\ -\frac{1}{2}(j_3^+ + j_2^+) + j_1^- & -\frac{1}{2}(j_1^+ + j_3^+) + j_2^- & -\frac{1}{2}(j_2^+ + j_1^+) + j_3^- \end{pmatrix}. \qquad \mathcal{R}_5 \; (3.8)$$

3.1. Features of $(3\text{-}j)$ symbols generated by Regge transformations

We will use various definitions and notations explicated below.

$(3\text{-}j) \xrightarrow{S_p} \{(3\text{-}j)\} =$ a set denoted by S_p that contains twelve $(3\text{-}j)$.

$(3\text{-}j) \in S_p \xrightarrow{\mathcal{R}_\kappa} (3\text{-}j)^{\mathcal{R}_\kappa} \in S_p^{\mathcal{R}_\kappa}, \; \kappa \in [1, 5].$

Let be $n_\varnothing$ the number of empty intersections satisfying to

$$S_p^{\mathcal{R}_\kappa} \cap S_P^{\mathcal{R}_\lambda} \cap S_p = \varnothing, \quad \kappa \neq \lambda \in [1, 5]. \qquad (3.9)$$

A *priori* it results that 6 disjoint sets $S_p(n_\varnothing)$ may be defined for $n_\varnothing \in [0, 5]$.

If a set $S_p(n_\varnothing)$ is not empty, then it contains $12(n_\varnothing + 1)$ $(3\text{-}j)$ symbols.

Filtering operation $\left(S_p \text{ filter}\right)$:

$\mathcal{R}_{all}$ denotes the five Regge transformations. $\mathcal{R}_{all}$ applied to a $(3\text{-}j)_0$ yields a list

$$\mathcal{R}_{all}((3\text{-}j)_0) = \{(3\text{-}j)^{\mathcal{R}_1}, (3\text{-}j)^{\mathcal{R}_2}, (3\text{-}j)^{\mathcal{R}_3}, (3\text{-}j)^{\mathcal{R}_4}, (3\text{-}j)^{\mathcal{R}_5}\}. \tag{3.10}$$

If $(3\text{-}j)^{\mathcal{R}_\lambda} \in S_p^{\mathcal{R}_\kappa}$, $\lambda \neq \kappa \in [1, 5];$, then $(3\text{-}j)^{\mathcal{R}_\lambda}$ is deleted from the list (3.10). After this first operation there may remain at least *one* and at most *five* $(3\text{-}j)$ inside the list. Among the remaining $(3\text{-}j)$'s, we continue a similar operation by checking if a $(3\text{-}j) \in S_{p_0}$, if it is the case the $(3\text{-}j)$ is deleted from the remaining list. It may happen that the final list is empty. The operation described above is denoted by $\left(S_p \text{ filter}\right)$ and we define $\mathcal{R}^*_{egge}$ by

$$\mathcal{R}^*_{egge} = (S_p \text{ filter}) \circ \mathcal{R}_{all}. \tag{3.11}$$

This allows us to build a partition of any $(3\text{-}j)$ symbols into $S_p(n_\varnothing)$ sets.

Closure property under $\mathcal{R}^*_{egge}$ is ensured namely $\mathcal{R}^*_{egge}(S_p(n_\varnothing)) \equiv S_p(n_\varnothing)$.

$$\tag{3.12}$$

The method is similar to that followed in our previous paper [6] about $\{6\text{-}j\}$.

Definitions. [(circ) will denote a circular permutation of (1, 2, 3)].

$$\begin{aligned} N_0^d \; = \; & \text{number of zeros of } \{(j_i^+ - j_k^+)_{i \neq k}\} \\ & + \text{number of zeros of } \{(j_i^- - j_k^-)_{i \neq k}\}. \end{aligned} \tag{3.13}$$

$$\begin{aligned} N_0^\pm \; = \; & \text{number of zeros of } \{(j_i^+ - j_k^-)_{i \neq k}\} \\ & + \text{number of zeros of } \{(j_i^- - j_k^+)_{i \neq k}\}. \end{aligned} \tag{3.14}$$

$$\begin{aligned} N_0^m \; = \; & \text{number of zeros of } \{(j_i^+ - j_i^-)\} \\ & \equiv \{(2m_i)\}, \text{ with values } 0, 1 \text{ or } 3. \end{aligned} \tag{3.15}$$

Consider 6 differences between the first row of the Regge array and the second or third.

$$\delta R_i^k = (R_1^k - R_i^k) \text{ with } i \in [2, 3],\ k \in [1, 3]. \tag{3.16}$$

Each quantity is a difference between a $(j^+ - j^-)$ and a $(j^- - j^+)$ or a $(j^+ - j^+)$:

$$\delta R_2^1 = (j_2^+ - j_1^-) - (j_1^- - j_3^+),\ \delta R_3^1 = (j_2^+ - j_1^-) - (j_1^+ - j_+^+) \text{ and so on.} \tag{3.17}$$

$$N_0^R = \text{number of zeros of } [\delta R_2] + \text{ number of zeros of } [\delta R_3]. \tag{3.18}$$

The partition selectors belong to a set $\mathcal{E}_{\text{Sel}}$ (15 elements) defined by

$$\mathcal{E}_{\text{Sel}} = \left\{ \begin{pmatrix} \#{=}3 \\ j_i^+ - j_k^+ \end{pmatrix}, \begin{pmatrix} \#{=}3 \\ j_i^- - j_k^- \end{pmatrix}, \begin{pmatrix} \#{=}3 \\ j_i^+ - j_k^- \end{pmatrix}, \begin{pmatrix} \#{=}3 \\ j_i^- - j_k^+ \end{pmatrix}, \begin{pmatrix} \#{=}3 \\ j_i^+ - j_i^- \end{pmatrix}, \right\}_{i \neq k} i, k \in [1, 3]. \tag{3.19}$$

As $\begin{bmatrix} \#{=}3 \\ \delta R_2 \end{bmatrix}, \begin{bmatrix} \#{=}3 \\ \delta R_3 \end{bmatrix}$ are linear combinations of elements of $\mathcal{E}_{\text{Sel}}$, they are not accounted for.

The partitions and selectors found are shown below.

$$S_p(0) = \{(3\text{-}j) |\ N_0^{\pm} \in [3, 4] \text{ or } N_0^{\pm} = 6\}, \tag{3.20}$$

$$S_p(1) = \{(3 - j) |\ N_0^{\pm} = 2\}, \tag{3.21}$$

$$S_p(2) = \{(3 - j) |\ N_0^{\pm} = 1\}, \tag{3.22}$$

$$S_p(3) = \varnothing, \tag{3.23}$$

$$S_p(4) = \{(3 - j) |\ N_0^{\pm} = 0,\ (N_0^m = 0) \text{ and}$$

$$N_0^d = 2,\ N_0^R = 0,\ (((j_1^+ = j_2^+) \text{ and } (j_1^- = j_2^-)) \text{ or } (\text{circ}))$$

or

$$N_0^d = 0,\ N_0^R = 3) \text{ or } (N_0^d = 0,\ N_0^R = 0))$$

$$\oplus$$

$$N_0^{\pm} = 0,\ (N_0^m = 1) \text{ and } N_0^d = 0,\ (N_0^R = 4)$$

$$\oplus$$

$$N_0^{\pm} = 0,\ (N_0^m = 3) \text{ and } (N_0^d = 0,\ N_0^R = 0 \text{ or } 2)\}, \quad (3.24)$$

$$S_p(5) = \{(3-j)|\ N_0^{\pm} = 0,\ (N_0^m = 0) \text{ and}$$

$$N_0^d = 2,\ N_0^R = 0,\ (((j_1^+ = j_2^+) \text{ and } (j_1^- \neq j_2^-)) \text{ or } (\text{circ}))$$

or

$$(N_0^d \in [0,1],\ N_0^R \in [0,2]) \text{ or } (N_0^d = 3,\ N_0^R = 0)$$

$$\oplus$$

$$N_0^{\pm} = 0,\ (N_0^m = 1) \text{ and}$$

$$(N_0^d = 0,\ N_0^R \in [0,2]) \text{ or } (N_0^d = 1,\ N_0^R \in [0,1])\}. \quad (3.25)$$

Instead of 4 for $\{6\text{-}j\}$ symbols, we find here 5 partitions for $(3\text{-}j)$ symbols.

A symbolic sequence illustrates the results where over each subset is indicated its cardinal:

$$(3-j) \overset{*}{+} \text{Regge symmetry} \rightarrow \overset{\#=12}{S_p(0)} \oplus \overset{\#=24}{S_p(1)} \oplus \overset{\#=36}{S_p(2)} \oplus \overset{\#=60}{S_p(4)} \oplus \overset{\#=72}{S_p(5)}\ . \quad (3.26)$$

As expected the larger symmetry group of order 72, i.e., $\overset{\#=72}{S_p(5)}$, is well retrieved, however, what remained unknown up to today is the existence of intermediate groups of order 12, 24, 36, 60 with exclusion of the order 48 for $S_p(3)$.

Achieving this difficult classification requires some comment. Our former

Fortran 95 program (*symmetryregge*) [6] has been modified into (*supersymbol3jcount*) where this time a comparison of a lot of S_p sets is carried out. The discoveries of partition selectors are not automatic. Only a thorough examination, logical or intuitive, allows one to find them. For lack of a formal logic program able to optimize or reduce possible redundancies, we can not assert that our selectors are the best. Nevertheless, what is irrefutable is the existence of partitions and selectors. It may be noted also that our results are purely 'computed' and do not derive from a group-theoretical analysis.

4. Analytic Formula for Super $(3\text{-}j)^S$ Symbols

For beginning we need to find a formula for these supersymmetric $(3\text{-}j)^S$ symbols the most similar to that of standard $(3\text{-}j)$ symbols. Let us start by updating definitions used in an ancient paper [8].

$$\Delta^S(abc) = \left(\frac{[a+b-c]!\,[a-b+c]!\,[-a+b+c]!}{\left[a+b+c+\dfrac{1}{2}\right]!} \right)^{\frac{1}{2}} \qquad \text{(supertriangle)}. \quad (4.1)$$

Delimiters $[]$ around a *number*, integer or half-integer, mean 'integer part of *number*'. ∇ stands for Δ^{-1} and ∇^S for $(\Delta^S)^{-1}$.

A (so-called) parity independent $(3\text{-}j)^S$ symbol[2] was introduced by Daumens et al. [7] as the product of a scalar factor by a standard $(3\text{-}j)$ symbol [its $\mathfrak{su}(2)$ 'parent']:

$$\begin{pmatrix} j_i & j_2 & j_3 \\ l_1 m_1 & l_2 m_2 & l_3 m_3 \end{pmatrix} = \begin{bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{bmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \qquad (4.2)$$

We have proved [8] that any scalar factor can be written as:

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[2]denoted in [7] by $S3\text{-}j$.

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{bmatrix} = (-1)^{\phi_{[]}} \begin{cases} \nabla(l_1 l_2 l_3)\,\Delta^S(j_1 j_2 j_3) & j_1+j_2+j_3 \text{ integer}, \\ \Delta(l_1 l_2 l_3)\,\nabla^S(j_1 j_2 j_3) & j_1+j_2+j_3 \text{ half-integer}, \end{cases} \tag{4.3}$$

where the general phase factor $\phi_{[]}$ can be rewritten as

$$(-1)^{\phi_{[]}} = (-1)^{2(j_1+j_2+j_3)+8(j_1-l_1)(j_2-l_2)(j_3-l_3)+4(l_1(j_3+l_3)+l_2(j_1+l_1)+l_3(j_2+l_2))}. \tag{4.4}$$

We will reuse also our shortened notation of a $(3\text{-}j)^S$, which drops out all l's:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^S = \begin{pmatrix} j_1 & j_2 & j_3 \\ l_1 m_1 & l_2 m_2 & l_3 m_3 \end{pmatrix}. \tag{4.5}$$

It must be realized that the definition of a super $(3\text{-}j)^S$ implies two triangular constraints, one for the triangle $(j_1\ j_2\ j_3)$ with integer or half-integer perimeter, the other for $(l_1\ l_2\ l_3)$ with integer perimeter only. For example, $|j_1 - j_2| \le j_3 \le j_1 + j_2$ and $|l_1 - l_2| \le l_3 \le l_1 + l_2$.

It is important for establishing a correct table of $(3\text{-}j)^S$ symbols from (4.2), (4.5) where the l's are no more visible and spins j are incremented by step of $\frac{1}{2}$. While forgetting the condition on the l's, we might have to compute a super symbol like $\begin{pmatrix} 7/2 & 2 & 3/2 \\ -1/2 & 1/2 & 0 \end{pmatrix}^S$, that has no existence because its parent $\begin{pmatrix} 7/2 & 3/2 & 1 \\ -1/2 & 1/2 & 0 \end{pmatrix}$ is not a valid $(3\text{-}j)$ symbol for $\mathfrak{su}(2)$. For the calculations now, it seems judicious to gather some square roots together and define a super scalar factor as

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix} = \Delta(l_1\ l_2\ l_3) \begin{bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{bmatrix} \text{ (super scalar factor).} \tag{4.6}$$

It is of interest because this super-factor then depends simply on an integer positive $I(j_1\ j_2\ j_3)$ and on $(j_k \pm m_k)$ for the phase. The result reads

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix}^S = (-1)^{\phi[]} \Delta^S (j_1\ j_2\ j_3) I(j_1\ j_2\ j_3), \tag{4.7}$$

$$I(j_1\ j_2\ j_3) = 1 \quad \text{if} \quad j_1 + j_2 + j_3 = \text{integer}, \tag{4.8}$$

$$I(j_1\ j_2\ j_3) = \sum_k (|(-1)^{2(j_k - m_k)} j_k|) + \frac{1}{2} \quad \text{if} \quad j_1 + j_2 + j_3 = \text{half-integer}. \tag{4.9}$$

Expression $\sum_k$ is a trick for representing the four possible positive integer values of I:

$$I_1 = \left(- j_1 + j_2 + j_3 + \frac{1}{2}\right), \quad I_2 = \left(j_1 - j_2 + j_3 + \frac{1}{2}\right),$$

$$I_3 = \left(j_1 + j_2 - j_3 + \frac{1}{2}\right), \quad I_4 = \left(j_1 + j_2 + j_3 + \frac{1}{2}\right). \tag{4.10}$$

Another trick unifying (4.8)-(4.9) into a single formula is the use of integer parts and factorial.

$$I(j_1\ j_2\ j_3) = \frac{\left[\left|\sum_k (-1)^{2(j_k - m_k)} j_k\right| + \frac{1}{2}\right]!}{\left[\left|\sum_k (-1)^{2(j_k - m_k)} j_k\right|\right]!}. \tag{4.11}$$

An essential remark concerns the possible doublets $l_k = j_k$, $l_k = j_k - \frac{1}{2}$. We have

$$(l_k \pm m_k) = [j_k \pm m_k] = [j_k^{\pm}]. \tag{4.12}$$

This gives the means to end all rearrangements and adopt a definition of a $(3\text{-}j)^S$ symbol fully similar to that of a $(3\text{-}j)$ symbol in three equations like (2.1)-(2.2)-(2.3).

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^S = \Delta^S (j_1\ j_2\ j_3) v^S \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{4.13}$$

$$\Delta^S(j_1\ j_2\ j_3) = \left(\frac{[j_1 + j_2 - j_3]![j_1 - j_2 + j_3]![-j_1 + j_2 + j_3]!}{[j_1 + j_2 + j_3 + \frac{1}{2}]!}\right)^{\frac{1}{2}}, \tag{4.14}$$

$$v^S\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{[j_1^+]-[j_2^-]+\sum_k 2j_k + 8\prod_k j_k^{\pm} + 4(j_1^{\pm}m_2 + j_2^{\pm}m_3 + j_3^{\pm}m_1)}$$

$$\times\left(\prod\nolimits_k [j_k^+]![j_k^-]!\right)^{\frac{1}{2}} \frac{\left[\left|\sum_k (-1)^{2j_k^{\pm}} j_k\right| + \frac{1}{2}\right]!}{\left[\left|\sum_k (-1)^{2j_k^{\pm}} j_k\right|\right]!}$$

$$\times\sum_z \frac{(-1)^z}{z!(z-([j_2^+]-[j_3^-]))!(z-([j_1^-]-[j_3^+]))!(([j_1^+]+[j_2^+]-[j_3^-])-z)!([j_1^-]-z)!([j_2^+]-z)!}. \tag{4.15}$$

For each $(3\text{-}j)^S$, $\mathfrak{su}(2)$ isospin doublets can be retrieved by using $2l_k = [j_k^+] + [j_k^-]$. Expressions (4.13)-(4.15) allows one to compute a large table of $(3\text{-}j)^S$ that fits with analytic formulas (where one spin equals $\frac{1}{2}$) given in [7], after the correction of a misprint[3].

5. Regge Symmetry of $(3\text{-}j)^S$ Symbols

Exactly as for $\{6\text{-}j\}^S$ [6], it is found that $(3\text{-}j)^S$ symbols admit a classification with three intrinsic parities which we will call again α, β, γ without confusion with the former ones.

Parity α: $\quad (\overset{|ev|}{c_1}\ \overset{|ev|}{c_2}\ \overset{|ev|}{c_3})^S_\alpha.$

[3][7], p. 2495, Table IV- Analytic values of $S3\text{-}j$ symbols, third formula: $\sqrt{\frac{1}{2}}$ to be removed.

Parity β: $\begin{pmatrix} |ev| & |od| & |od| \\ c_1 & c_2 & c_3 \end{pmatrix}^S_{\beta_1}$ or $\begin{pmatrix} |od| & |ev| & |ev| \\ c_1 & c_2 & c_3 \end{pmatrix}^S_{\beta_1'}$,

$\begin{pmatrix} |od| & |ev| & |od| \\ c_1 & c_2 & c_3 \end{pmatrix}^S_{\beta_2}$ or $\begin{pmatrix} |ev| & |od| & |ev| \\ c_1 & c_2 & c_3 \end{pmatrix}^S_{\beta_2'}$,

$\begin{pmatrix} |od| & |od| & |ev| \\ c_1 & c_2 & c_3 \end{pmatrix}^S_{\beta_3}$ or $\begin{pmatrix} |ev| & |ev| & |od| \\ c_1 & c_2 & c_3 \end{pmatrix}^S_{\beta_3'}$.

Parity γ: $\begin{pmatrix} |od| & |od| & |od| \\ c_1 & c_2 & c_3 \end{pmatrix}^S_{\gamma}$.

Parity α contains only $j_1 + j_2 + j_3$ integer, β can contain $j_1 + j_2 + j_3$ integer (β_κ) or half-integer (β'_κ) and γ only $j_1 + j_2 + j_3$ half-integer. Actually this discrepancy is embedded via the analytic expression of $I(j_1\ j_2\ j_3)$ given by (4.11), so that a best classification of $(3\text{-}j)^S$ symbols should be expressed in terms of 'column-parity' and no longer by dichotomizing the cases where $\sum_k j_k$ is integer or half-integer.

According to our defining choice of Regge transformations $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, $\mathcal{R}_4$, $\mathcal{R}_5$, note that I_1 is invariant only under $\mathcal{R}_1$ (parity β_1, β_1'), I_2 only under $\mathcal{R}_2$ (parity β_2, β_2'), I_3 only under $\mathcal{R}_3$ (parity β_3, β_3'), and I_4 under $\mathcal{R}_{all}$ (parity γ). I numbers were defined by (4.10).

A quick reading of the Regge transformations [such as they have been written by (3.4)-(3.8)] indicates right away what are the symbols possessing a (super) Regge symmetry.

5.1. Features of $(3\text{-}j)^S$ symbols generated by Regge transformations

Parity α, γ:

In this case properties like (3.20)-(3.25) of course are still valid. Thus analogously

$$(3\text{-}j)^S_{\alpha,\gamma} + \overset{*}{\text{R}}\text{egge symmetry} \to \overset{\#=12}{S^S_p(0)} \oplus \overset{\#=24}{S^S_p(1)} \oplus \overset{\#=36}{S^S_p(2)} \oplus \overset{\#=60}{S^S_p(4)} \oplus \overset{\#=72}{S^S_p(5)}. \quad (5.1)$$

Parity β: (Indices $\kappa \in [1, 3]$ of β_κ, β'_κ are no longer significant)

Only two sets may exist, namely $S_p^S(0)$ and $S_p^S(1)$ defined by the selector $N_0^\pm$:

$$S_p^S(0) = \{(3\text{-}j)_\beta^S\}| \quad N_0^\pm \in [1, 2], \tag{5.2}$$

$$S_p^S(1) = \{(3\text{-}j)_\beta^S\}| \quad N_0^\pm = 0. \tag{5.3}$$

The analog of (5.1) then becomes

$$(3\text{-}j)_\beta^S + \overset{*}{\text{R}}\text{egge symmetry} \to \overset{\#=12}{S_p^S(0)} \oplus \overset{\#=24}{S_p^S(1)}. \tag{5.4}$$

Moreover, an unexpected specificity of β parity regards the sign of the numerical values of a symbol $(3\text{-}j)_\beta^S$ and its transformed by Regge: it can be $\pm$.

This is explainable by the following proof: Regge transformations such as described by (3.4)-(3.8) and applied formally to a $(3\text{-}j)^S$ leave invariant $\sum_k 2j_k$, $\forall$ transformation $(3\text{-}j)^S \xrightarrow{\mathcal{R}_\kappa} (3\text{-}j')^S$ with $\kappa \in [1, 5]$. It can be proved that only two phases are relevant:

$$(-1)^{\phi^S} = -(-1)^{8\prod_k j_k^\pm + 4(j_1^\pm m_2 + j_2^\pm m_3 + j_3^\pm m_1)}. \tag{5.5}$$

$$(-1)^{\phi'^S} = -(-1)^{8\prod_k j_k'^\pm + 4(j_1'^\pm m_2' + j_2'^\pm m_3' + j_3'^\pm m_1')}. \tag{5.6}$$

From (4.2) it can be seen that

$$(3\text{-}j')^S = (-1)^{\phi^S + \phi'^S} \times (3\text{-}j)^S. \tag{5.7}$$

For parities α, γ, we have $(-1)^{\phi^S + \phi'^S} = +1$.

Consider a $\mathcal{R}_1$ transformation, valid for a $(3\text{-}j)_{\beta_1}^S$, we find a phase $(-1)^{\phi_{\mathcal{R}_1}^S(c_1 c_2 c_3)}$ given by

$$(-1)^{\phi^S_{\mathcal{R}_1}(c_1\, c_2\, c_3)} = (-1)^{\phi^S_{\beta_1} + \phi'^S_{\beta_1}}$$

$$= (-1)^{2j_1 + 4j_1 m_1 + 2j_1^+(j_2^+ - j_3^+) + ((\Sigma_k 2j_k) + 1)(j_3^- - j_2^- + 1) + 2m_2 + 1}. \tag{5.8}$$

From our definitions of $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, it is clear that

$$\phi^S_{\mathcal{R}_2}(c_1\, c_2\, c_3) = \phi^S_{\mathcal{R}_1}(c_2\, c_1\, c_3) \quad \text{and} \quad \phi^S_{\mathcal{R}_3}(c_1\, c_2\, c_3) = \phi^S_{\mathcal{R}_2}(c_1\, c_3\, c_2). \tag{5.9}$$

In shortcut

$$(3\text{-}j)^S_{\beta_\kappa} \xrightarrow{\mathcal{R}_\kappa} (3\text{-}j')^S_{\beta_\kappa} \Rightarrow (3\text{-}j')^S_{\beta_\kappa} = (-1)^{\phi^S_{\mathcal{R}_\kappa}} \times (3\text{-}j)^S_{\beta_\kappa} \text{ with } \kappa \in [1, 3]. \tag{5.10}$$

Accordingly, Regge transformations for β parity may bring a phase, or not. It depends if $\phi^S_{\mathcal{R}_\kappa}$ is even or odd. Tests on computer turn out satisfactory.

6. Analytic Prolongation of $(3\text{-}j)^S$ Symbols

An attempt for extrapolating our table of $(3\text{-}j)^S$ symbols to forbidden cases like $l_3 < |l_1 - l_2|$ or $l_3 > l_1 + l_2$ produces indefinite values, as expected. It shows that only cases of parity β are implicated with flat integer triangles defined by $j_\kappa = j_\lambda + j_\mu$, $(\kappa, \lambda, \mu) = \text{circ}(1, 2, 3)$. Let us denote these forbidden cases by $(3\text{-}j)^{S\times}_\beta$ [superscript $\times$ stands for 'forbidden']. So to say, they are 'orphans', i.e., without $\mathfrak{su}(2)$ parent. The meaning of scalar factors [ ] or integers I vanishes, *at first sight*. For a given κ, orphan symbols $(3\text{-}j)^{S\times}_\beta$ are precisely of the kind $(3\text{-}j)^{S\times}_{\beta_\kappa}$.

For example consider $\begin{pmatrix} |od| & |od| & |ev| \\ j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^{S\times}_{\beta_3}$, where $j_3 = j_1 + j_2$. Clearly m_1 can take values varying by a step of 1:

$m_1 = -j_1 + \dfrac{1}{2}, -j_1 + \dfrac{3}{2}, \cdots, j_1 - \dfrac{3}{2}, j_1 - \dfrac{1}{2}$. The same holds for m_2. The variation range of m_3 is similar, namely: $m_3 = -j_3 + 1, -j_3 + 2, \cdots, j_3 - 2, j_3 - 1$. Each increment is 1.

This leads immediately to an analogy with a standard (flat) symbol $(3\text{-}j)$ whose value is derived from a formula given by Edmonds [9, p. 48]. That reads

$$\begin{pmatrix} |ev| & |ev| & |ev| \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_1 + j_2 - 1 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{(j_1 - \frac{1}{2} + m_1) - (j_2 - \frac{1}{2} - m_2)}$$

$$\times \left[\frac{(2j_1 - 1)!(2j_2 - 1)!(j_1 + j_2 - 1 + m_1 + m_2)!(j_1 + j_2 - 1 - m_1 - m_2)!}{(2j_1 + 2j_2 - 1)!(j_1 - \frac{1}{2} + m_1)!(j_1 - \frac{1}{2} - m_1)!(j_2 - \frac{1}{2} + m_2)!(j_2 - \frac{1}{2} - m_2)!} \right]^{\frac{1}{2}}. \tag{6.1}$$

From (4.3), after noting that the scalar factor $\begin{bmatrix} j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 - 1 \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 - 1 \end{bmatrix} = [2j_3 - 1]^{\frac{1}{2}}$,

we re-write (6.1) under a form that highlights our proposal of analytic prolongation:

$$\begin{bmatrix} j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 - 1 \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 - 1 \end{bmatrix} \begin{pmatrix} j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_3 - 1 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{(j_1 - \frac{1}{2} + m_1) - (j_2 - \frac{1}{2} + m_2)}$$

$$\times \left[\frac{(2j_1 - 1)!(2j_2 - 1)!(j_3 - 1 + m_3)!(j_3 - 1 - m_3)!}{(2j_3 - 2)!(j_1 - \frac{1}{2} + m_1)!(j_1 - \frac{1}{2} - m_1)!(j_2 - \frac{1}{2} + m_2)!(j_2 - \frac{1}{2} - m_2)!} \right]^{\frac{1}{2}}. \tag{6.2}$$

Analytic prolongation definition:

In a way fully similar to (4.2), we adopt the following definition, with $j_1, j_2 \geq \frac{1}{2}, j_3 \geq 1$:

$$\begin{pmatrix} |od| & |od| & |ev| \\ j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^{S\times}_{\beta_3 \ |j_3=j_1+j_2} = \begin{bmatrix} j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_3-1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_3-1 \end{bmatrix} \begin{pmatrix} j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_3-1 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$= (-1)^{j_1^+ - j_2^-} \left[\frac{(j_3^+ -1)!(j_3^- -1)!}{(2j_3-2)!} \right]^{\frac{1}{2}} \left[\frac{(2j_1-1)!(2j_2-1)!}{[j_1^+]![j_1^-]![j_2^+]![j_2^-]!} \right]^{\frac{1}{2}}. \quad (6.3)$$

Then $(3\text{-}j)^{S\times}_{\beta_3 \ |\text{flat}}$ can be re-integrated in the set of regular $(3\text{-}j)^S$ symbols, according to a single set of equalities $l_1 = j_1 - \frac{1}{2}$, $l_2 = j_2 - \frac{1}{2}$, $l_3 = j_3 - 1$, by making the following identification

$$\begin{pmatrix} |od| & |od| & |ev| \\ j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^{S\times}_{\beta_3 \ |j_3=j_1+j_2} \approx \begin{pmatrix} |ev| & |ev| & |ev| \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_3-1 \\ m_1 & m_2 & m_3 \end{pmatrix}^{S}_{\alpha \ |j_3=j_1+j_2}. \quad (6.4)$$

More generally

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^{S\times}_{\beta_\kappa \ |\text{flat}} \approx \begin{pmatrix} j_\lambda-\frac{1}{2} & j_\mu-\frac{1}{2} & j_\kappa-1 \\ m_\lambda & m_\mu & m_\kappa \end{pmatrix}^{S}_{\alpha \ \Big| \substack{j_\kappa = j_\lambda + j_\mu \, j_\lambda,\, j_\mu \geq \frac{1}{2} \\ (\kappa,\lambda,\mu)=\text{circ}(1,2,3)}}, \quad (6.5)$$

with the following numerical value

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^{S\times}_{\beta_\kappa \ |\text{flat}}$$

$$= (-1)^{j_\lambda^+ - j_\mu^-} \left[\frac{(j_\kappa^+ -1)!(j_\kappa^- -1)!}{(2j_\kappa-2)!} \right]^{\frac{1}{2}} \left[\frac{(2j_\lambda-1)!(2j_\mu-1)!}{[j_\lambda^+]![j_\lambda^-]![j_\mu^+]![j_\mu^+]!} \right]^{\frac{1}{2}}. \quad (6.6)$$

Regge transformations and notation for flat triangles

Since a symbol $(3\text{-}j)^{S\times}_{\beta_\kappa}|_{\text{flat}}$ is actually of the kind α then $\mathcal{R}_{all}((3\text{-}j)^{S\times}_{\beta_\kappa}|_{\text{flat}})$ have their five identical numerical values, phase included. In order to ensure the closure property (3.12), we need an additional filtering operation $(S_p\ \underline{\text{filter}})$, where the bar which underlines means that only flat triangles $(j_1\ j_2\ j_3)$ are retained. Extension of this underlining will be used elsewhere with an obvious signification. Analogously to (3.11), we may define a $\underline{\mathcal{R}}^*_{egge}$ as

$$\underline{\mathcal{R}}^*_{egge} = (S_p\ \underline{\text{filter}}) \circ (S_p\ \text{filter}) \circ \mathcal{R}_{all}. \tag{6.7}$$

Clearly the number of disjoint sets $\underline{S}^S_p(n_\varnothing)$ will be reduced. A bit like for a true parity β (i.e., valid), the remaining selection comes from only one $\mathcal{R}_1$, or $\mathcal{R}_2$ or $\mathcal{R}_3$. Accordingly, both possible values of $n_\varnothing$ belong to the range [0, 1]. We can present the results as follows:

$$(3\text{-}j)^{S\times}_{\underline{\beta}} \approx (3\text{-}j)^S_{\underline{\alpha}} + \underline{\text{R}}^*\text{egge symmetry} \rightarrow \overset{\#=12}{\underline{S}^S_p(0)} \oplus \overset{\#=24}{\underline{S}^S_p(1)}\ . \tag{6.8}$$

The relevant selectors here and their notations are slightly different from (3.13)-(3.14).

$$\underline{N}^d_0 = \text{ number of zeros of } \{(j^+_i - j^+_k)_{i\neq k}\}$$

$$+ \text{ number of zeros of } \{(j^-_i - j^-_k)_{i\neq k}\}, \tag{6.9}$$

$$\underline{N}^\pm_0 = \text{ number of zeros of } \{(j^+_i - j^-_k)_{i\neq k}\}$$

$$+ \text{ number of zeros of } \{(j^-_i - j^+_k)_{i\neq k}\}, \tag{6.10}$$

where the spins j are re-defined from (6.5) by

$$j_\lambda = j_\lambda - \frac{1}{2}, \quad j_\mu = j_\mu - \frac{1}{2}, \quad j_\kappa = j_\kappa - 1. \tag{6.11}$$

Below are listed the selectors and their values such as we found them:

$$\underline{S}_p^S(0) = \{(3\text{-}j)^{S\times}_{\underline{\beta\kappa}} |\ \underline{N}_0^{\pm} = 1,\ \underline{N}_0^{d} \in [0, 2],\ ((j_\lambda^+ = j_\mu^-) \text{ or } (j_\lambda^- = j_\mu^+))$$

$$\text{or } [\underline{N}_0^{\pm} = 2],\ (\underline{N}_0^{d} = 0 \text{ or } 2),\ (j_\lambda^+ = j_\mu^-) \text{ and } (j_\lambda^- = j_\mu^+)$$

$$\text{or } [\underline{N}_0^{d} = 1 \text{ or } 3],\ ((j_\lambda^+ = j_\mu^- = j_\kappa^+) \text{ or } (j_\lambda^- = j_\mu^+ = j_\kappa^-))$$

$$\text{or } \underline{N}_0^{\pm} \in [3, 4] \text{ or } \underline{N}_0^{\pm} = 6\}, \tag{6.12}$$

$$\underline{S}_p^S(1) = \{(3\text{-}j)^{S\times}_{\underline{\beta\kappa}} |\ \underline{N}_0^{\pm} = 0 \text{ or } \underline{N}_0^{\pm} = 1,\ \underline{N}_0^{d} \in [0, 2] \text{ and}$$

$$((j_\lambda^+ = j_\kappa^-) \text{ or } (j_\lambda^+ = j_\kappa^+) \text{ or } (j_\mu^+ = j_\kappa^-) \text{ or } (j_\mu^- = j_\kappa^+) \text{ or}$$

$$[\underline{N}_0^{\pm} = 2],\ \underline{N}_0^{d} = 0,\ ((j_\lambda^+ = j_\kappa^-) \text{ and } (j_\lambda^- = j_\kappa^+)) \text{ or } ((j_\mu^+ = j_\kappa^-)$$

$$\text{and } (j_\mu^- = j_\kappa^+)) \text{ or } \underline{N}_0^{d} = 1,\ ((j_\lambda^+ = j_\mu^+ = j_\kappa^-)$$

$$\text{or } (j_\lambda^- = j_\mu^- = j_\kappa^+)) \text{ or } \underline{N}_0^{d} = 2,\ ((j_\lambda^+ = j_\kappa^-) \text{ and}$$

$$(j_\lambda^- = j_\kappa^+)) \text{ or } ((j_\mu^+ = j_\kappa^-) \text{ and } (j_\mu^- = j_\kappa^+))$$

$$\text{or} ((j_\lambda^+ = j_\mu^+ = j_\kappa^-) \text{ or } (j_\lambda^- = j_\mu^- = j_\kappa^+))\}. \tag{6.13}$$

Again, it is awkward to have so many defining equations of selectors for a few flat triangles. All could certainly be simplified and presented otherwise by optimizing the selectors that we have adopted throughout the research.

The advantage of the proposed analytical extension allows one to compute a complete table of $(3\text{-}j)^S$ symbols where the spins can vary by step of $\frac{1}{2}$ by considering only the triangular constraint on the triangle $(j_1\ j_2\ j_3)$. Some tables may be available on request, as text files, given as a listing similar to that used for our $\{6\text{-}j\}^S$ tables [6].

7. Conclusion

As known the set of σ-orbits provides a partition of any symmetric group S_k, however, the present situation is different since a partition of any $(3\text{-}j)$ or $(3\text{-}j)^S$ symbol is built from functional transformations (Regge). Although the $(3\text{-}j)$ symbols are objects simpler than the $\{6\text{-}j\}$ symbols [11], the partition selectors found here are much more complicated than those obtained in our previous study on $\{6\text{-}j\}/\{6\text{-}j\}^S$ [6]. This disparity has no explanation for now. Certainly recent innovative papers lead us discover another crucial role of Regge symmetries related to discretized quantum gravity models [12, 13]. However, the symmetries in question concern the symbols $\{6\text{-}j\}$ and not the $(3\text{-}j)$. Our analyses seem beyond the areas of all studies published on the topic, to date.

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