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# ONE PIECE

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**Abstract**

At first, we tile the plane by 8-gons. Then we present a way to tile the plane by $k$-gons for a every fixed $k$ for all natural numbers $k$ larger than two. We use an infinite number of equal tiles to cover the plane.

## 1. Introduction

It is a widespread opinion that one can tile the plane $\mathbb{R}^2$ only with triangles, squares and regular 6-gons. This is wrong. Here we show another possibility.

**Proposition 1.** *There is a tiling of the plane by* 8*-gons.*

**Proof.** Instead of a written proof we prefer to show a picture. See Figure 1.

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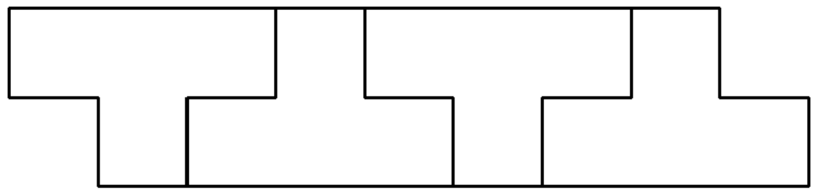

**Figure 1.** We see four equal 8-gons. With infinite many of these tiles we can cover the plane.

We think that it is useful to repeat the definition of a *simple polygon*.

A simple polygon with $k$ vertices consists of $k$ different points of the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_{k-1}, y_{k-1}), (x_k, y_k)$, called *vertices*, and the straight lines between $(x_i, y_i)$ and $(x_{i+1}, y_{i+1})$ for $1 \leq i \leq k-1$, called *edges*. Also the straight line between $(x_k, y_k)$ and $(x_1, y_1)$ belongs to the polygon. We demand that it is homeomorphic to a circle, and that there are no three consecutive collinear points $(x_i, y_i)$, $(x_{i+1}, y_{i+1})$, $(x_{i+2}, y_{i+2})$ for $1 \leq i \leq k-2$. Also the three points $(x_k, y_k)$, $(x_1, y_1)$, $(x_2, y_2)$ and $(x_{k-1}, y_{k-1})$, $(x_k, y_k)$, $(x_1, y_1)$ are not collinear.

We call this just described simple polygon a $k$*-gon*.

**Definition 1.** We call a polygon a *piece* if and only if it is one half of a regular 6-gon. Please see the picture Figure 2.

We use the word *doublepiece* as a synonym for a regular 6-gon. See Figure 2, too.

**Definition 2.** Let $t$ be any natural number. We call a polygon a $t$ row piece, if and only if $t$ pieces are put in a row. Two pieces are joined together at a common edge.

We call a polygon a $t$ row doublepiece, if and only if $t$ doublepieces are put in a row.

We call a polygon a $t$ shift square, if and only if $t$ squares are put upon the other, where the squares have side length 1 and each square is shifted by $\frac{1}{2}$. (Every number between 0 and 1 also works.)

Note that they are simple polygons.

See the example in Figure 3. There we show a 2 row piece and a 3 row doublepiece. See also the 3 shift square in Figure 4.

Note that a 1 row piece is just a piece, and a 1 row doublepiece is a doublepiece and also a regular 6-gon, and a 1 shift square is a square.

**Proposition 2.** *One can tile the plane with infinite copies of a $t$ row piece for all natural numbers $t$; also we can tile the plane with infinite copies of a $t$ row doublepiece for all $t$. Also we can tile the plane with infinite copies of a $t$ shift square for all $t$.*

**Proof.** Nearly trivial.

**Proposition 3.** *A $t$ row piece has $5 + 2(t - 1)$ vertices. A $t$ row doublepiece has $2 + 4t$ vertices. A $t$ shift square has $4t$ vertices.*

**Proof.** Easy. The proofs are by induction.

## 2. Tiling

**Theorem 1.** *There exists for all natural numbers $k$ larger than 2 a tiling of $\mathbb{R}^2$ with $k$-gons, where infinite copies of a single tile are used.*

**Proof.** For $k = 3$ and $k = 4$ and $k = 6$ the theorem is well-known. For $k = 5$ please see Figure 2. We use one piece and Proposition 2.

Now let $k$ be a natural number larger than 6.

**Lemma 1.** *It holds $k \equiv p \bmod 4$, where $p \in \{0, 1, 2, 3\}$.*

**Proof.** Well-known.

We discuss the four possibilities.

• Possibility 1: $p = 0$. In this case we get a suitable $t$ from the equation $k = 4t$ and we take a $t$ shift square as a $k$-gon. Please see Figure 4.

The sequence of the numbers of $k$ is 8, 12, 16, ... .

• Possibility 2: $p = 1$. These numbers are odd. Note that the set $\{5 + 2(t - 1) \mid t \in \mathbb{N}\}$ contains all odd numbers larger than 3. See Proposition 3. We take a suitable $t$ row piece as a $k$-gon. We get $t$ from the equation $k = 4t + 1$.

The sequence of the numbers of $k$ is 9, 13, 17, ... .

• Possibility 3: $p = 2$. We get $t$ from $k = 4t + 2$. We take a suitable $t$ row doublepiece as a $k$-gon.

The sequence of the numbers of $k$ is 10, 14, 18, ... .

• Possibility 4: $p = 3$. These numbers also are odd.

The sequence of the numbers of $k$ is 7, 11, 15, 19, ... .

The theorem is proven.

It follows some figures.

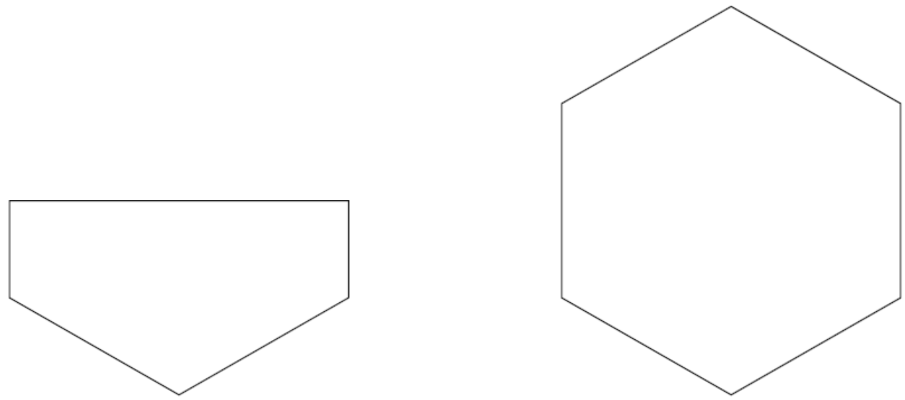

**Figure 2.** We see a piece and a doublepiece.

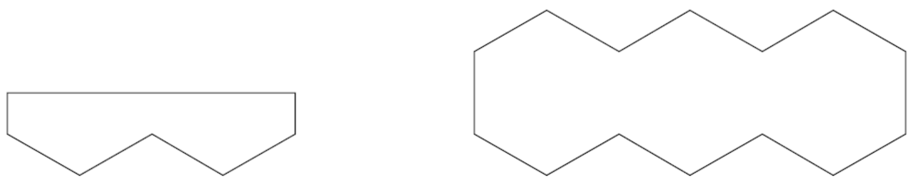

**Figure 3.** We see a 2 row piece and a 3 row doublepiece.

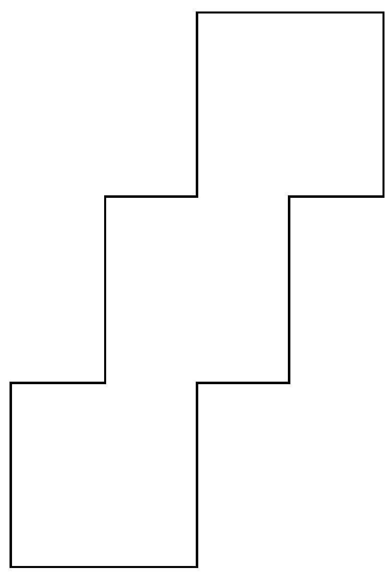

**Figure 4.** We see a 3 shift square.

### Acknowledgements

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For additional information, see the following references.

### References

[1] Ehrhard Behrends, Parkettierungen der Ebene, Springer, 2019.

[2] http://www.willimann.org/A07020-Parkettierungen-Theorie.pdf

[3] http://www.mathematische-Basteleien.de/parkett2.htm