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ON THE MATRIX REALIZATION OF THE LIE SUPERALGEBRA OF CONTACT PROJECTIVE VECTOR FIELDS $\mathfrak{spo}(2l+2|n)$

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Abstract

In this paper, we show that the Lie superalgebra $\mathfrak{spo}(2l+2|n)$ is into the intersection of Lie superalgebra of contact vector fields $\mathcal{K}(2l+1|n)$ and the Lie superalgebra of projective vector fields $\mathfrak{pgl}(2l+2|n)$. We use mainly the embedding used by P. Mathonet and F. Radoux in "Projectively equivariant quantizations over superspace $\mathbb{R}^{p/q}$, Lett. Math. Phys. 98 (2011), 311-331". Explicitly, we use the embedding of a Lie superalgebra constituted of matrices belonging to $\mathfrak{gl}(2l+2|n)$ into $\text{Vect}(\mathbb{R}^{2l+1|n})$. We generalize thus in superdimension $2l+1-n$, the matrix realization described in [7] on $S^{1/2}$. We mention that the intersection $\mathfrak{spo}(2l+2|n) = \mathfrak{pgl}(2l+2|n) \cap \mathcal{K}(2l+1|n)$ that we prove here, in super case, has been proved on $\mathbb{R}^{2l+2}$ in even case in [4].

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1. Introduction

The present paper is based on the concepts of supergeometry. It begins with a brief introduction to the notions that we need in the all sections, i.e., superfunctions, vector fields, differential 1-superforms, etc., on the superspace $\mathbb{R}^{m|n}$, where m and n are integers. We describe the supergeometry of $\mathbb{R}^{m|n}$ by its supercommutative superalgebra of superfunctions $C^\infty(\mathbb{R}^{m|n})$.

Using the standard contact structure on $\mathbb{R}^{2l+1|n}$, where l is also an integer, we compute the formula of the contact vector fields on $\mathbb{R}^{2l+1|n}$. This formula is a generalization of those formulas known in classical geometry, as in [4] and in supergeometry in low dimensions, as in [9, 11, 7]. We also compute, in the super case, the formula of the Lagrange bracket of the superfunctions f and g.

As in [7], we consider an superskewsymmetric form ω defined on the superspace $\mathbb{R}^{2l+2|n}$ and we realize thus a Lie superalgebra $\mathfrak{spo}(2l+2|n)$ constituted by the matrices A of $\mathfrak{gl}(2l+2|n)$ which preserve the form ω. We use the method used by P. Mathonet and F. Radoux in [6]. This construction allows us to embed the Lie superalgebra $\mathfrak{spo}(2l+2|n) \subset \mathfrak{pgl}(2l+2|n)$ into the Lie superalgebra $\mathrm{Vect}(\mathbb{R}^{2l+1|n})$ of vector fields on $\mathbb{R}^{2l+1|n}$.

Thanks to the formula of contact vector fields X_f obtained, for a certain superfunction given $f \in C^\infty(\mathbb{R}^{2l+1|n})$ of degree to most equal two in z, x_i, y_i and θ_i variables, and to the formulas of projective vector fields of $\mathfrak{spo}(2l+2|n) \subset \mathfrak{pgl}(2l+2|n)$ obtained, we realize that the Lie superalgebra $\mathfrak{spo}(2l+2|n)$ is constituted by the contact projective vector fields, i.e., the Lie superalgebra $\mathfrak{spo}(2l+2|n)$ is into the intersection of the Lie superalgebra of projective vector fields $\mathfrak{pgl}(2l+2|n)$ and the Lie superalgebra of contact vector fields $\mathcal{K}(2l+1|n)$. To justify the terminology of contact projective vector fields for the elements of $\mathfrak{spo}(2l+2|n)$, we refer to [4].

2. Superfunctions on $\mathbb{R}^{2l+1|n}$

We define the geometry of the superspace $\mathbb{R}^{2l+1|n}$, where $l \in \mathbb{N}$, $n \in \mathbb{N}^*$, by describing its associative supercommutative superalgebra of superfunctions on $\mathbb{R}^{2l+1|n}$ which we denote by

$$C^\infty(\mathbb{R}^{2l+1|n}) := C^\infty(\mathbb{R}^{2l+1}) \otimes \Lambda\mathbb{R}^n$$

and which is constituted by the elements

$$\begin{aligned} f(x, \theta) &= \sum_{0 \le |I| \le n} f_I(x)\theta_I \\ &= f_0(x) + f_1(x)\theta_1 + \ldots + f_n(x)\theta_n + f_{12}(x)\theta_1\theta_2 + \ldots + f_{1\ldots n}(x)\theta_{1\ldots}\theta_n, \end{aligned}$$

where $|I|$ is the length of I, $x = (x_i)$, $i = 1, \cdots, 2l+1$ is a coordinates system on $\mathbb{R}^{2l+1}$ and where $\theta = (\theta_i)$, $i = 1, \cdots, n$ is odd Grassmann coordinates on $\Lambda\mathbb{R}^n$, i.e., $\theta_i^2 = 0$, $\theta_i\theta_j = -\theta_j\theta_i$. We define the parity function $\tilde{}$ by setting $\tilde{x} = 0$ and $\tilde{\theta} = 1$.

3. Vector Fields on $\mathbb{R}^{2l+1|n}$

A vector field on $\mathbb{R}^{2l+1|n}$ is a superderivation of the associative supercommutative superalgebra $C^\infty(\mathbb{R}^{2l+1|n})$. In coordinates, it can be expressed as

$$X = \sum_{i=1}^{2l+1} X^i \partial_{x_i} + \sum_{j=1}^{n} Y^j \partial_{\theta_j},$$

where X^i and Y^j are the elements of $C^\infty(\mathbb{R}^{2l+1|n})$, $\partial_{x_i} = \frac{\partial}{\partial x_i}$ and $\partial_{\theta_j} = \frac{\partial}{\partial \theta_j}$ for all $i = 1, 2, \cdots, 2l+1$ and $j = 1, 2, \ldots, n$.

It can also be expressed as

$$X = \sum_{i=1}^{p+q} X^i \partial_{z_i},$$

where $z_i = x_i$ for all $i \in \{1, ..., 2l+1\}$ and $z_i = \theta_{i-(2l+1)}$ for all $i \in \{2l+2, ..., 2l+1+n\}$. The parity function $\tilde{\cdot}$ on vector field X is defined as

$$\tilde{\partial}_{x_i} = 0 \text{ and } \tilde{\partial}_{\theta_i} = 1.$$

The superspace of all vector fields on $\mathbb{R}^{2l+1|n}$ is a Lie superalgebra, which we shall denote by $\mathrm{Vect}(\mathbb{R}^{2l+1|n})$, by defining the following Lie bracket

$$[X, Y] = XY - (-1)^{\tilde{X}\tilde{Y}} YX$$

for all vector fields X, Y.

4. Differential 1-superforms on $\mathbb{R}^{2l+1|n}$

We define the superspace $\Omega^1(\mathbb{R}^{2l+1|n})$ of differential 1-superforms on $\mathbb{R}^{2l+1|n}$ as a superspace which is constituted by the elements

$$\alpha = \sum_{i=1}^{2l+1} f_i(x_i, \theta_i) dx^i + \sum_{i=1}^{n} g_i(x_i, \theta_i) d\theta^i,$$

where f_i and g_i are elements of $C^\infty(\mathbb{R}^{2l+1|n})$ and $\widetilde{dx^i} = 0$, $\widetilde{d\theta^i} = 1$ and where we set $\mathcal{B}' = (dx^i, d\theta^i)$ of $\Omega^1(\mathbb{R}^{2l+1|n})$ the dual basis of a basis $\mathcal{B} = (\partial_{x_i}, \partial_{\theta_i})$ of $\mathrm{Vect}(\mathbb{R}^{2l+1|n})$ such that

$$\langle \partial_{x_j}, dx^i \rangle = \delta^i_j, \quad \langle \partial_{x_j}, d\theta^i \rangle = 0 \text{ and } \langle \partial_{\theta_j}, d\theta_i \rangle = -\delta^i_j.$$

Remark 4.1. These elements f_i and g_i can also be declared at right and in this case we must use the even sign rule known in supergeometry.

When we consider a vector field X, we can also define the evaluation of differential 1-superform on X, or the interior product of a differential 1-superform α by X as follows:

$$\alpha(X) = (-1)^{\tilde{X}\tilde{\alpha}}\langle X, \alpha\rangle \text{ and } i(X)\alpha = \langle X, \alpha\rangle.$$

Explicitly, if $X = \sum_{i=1}^{2l+1+n} X^i \partial_{z_i}$ and $\alpha = \sum_{j=1}^{2l+1+n} \alpha_j dz_j$, we have via the sign rule,

$$\langle X, \alpha\rangle = \left\langle \sum_{i=1}^{2l+n+1} X^i \partial_{z_i}, \sum_{j=1}^{2l+n+1} \alpha_j dz_j \right\rangle = \sum_{i,j=1}^{2l+n+1} X^i \alpha_j (-1)^{\tilde{i}\tilde{\alpha}_j} \langle \partial_{z_i}, dz_j\rangle$$

$$= \sum_{i=1}^{2l+n+1} (-1)^{\tilde{i}(\tilde{\alpha}_i + \tilde{i})} X^i \alpha_i. \tag{1}$$

We can generalize the definition of differential superforms and we have also a version of de de Rham differential which is adapted in the framework of supergeometry. Thus it allows us to define the Lie derivative of differential superforms. These operators have the analogue properties known in classical geometry.

5. Standard Contact Structure on $\mathbb{R}^{2l+1|n}$

We consider here the standard contact structure on $\mathbb{R}^{2l+1|n}$. We can find in [10] the notions of the contact structure on any supermanifold of dimension $m|n$.

Definition 5.1. The standard contact structure on $\mathbb{R}^{2l+1|n}$ is defined by the kernel of the differential 1-superforms α on $\mathbb{R}^{2l+1|n}$ which, in the system of Darboux coordinates (z, x_i, y_i, θ_j), $i = 1, \ldots, l$ and $j = 1, \ldots, n$ it can be written as

$$\alpha = dz + \sum_{i=1}^{l} (x_i dy_i - y_i dx_i) + \sum_{i=1}^{n} \theta_i d\theta_i. \tag{2}$$

This differential 1-superform α is called contact form on $\mathbb{R}^{2l+1|n}$ and we denote by

$\mathrm{Tan}(\mathbb{R}^{2l+1|n})$ the space constituted of the elements of the kernel of α.

If we denote $q^A = (z, q^r)$, the generalized coordinate where

$$q^A = \begin{cases} z \text{ if } A = 0, \\ x_A \text{ if } 1 \leq A \leq l, \\ y_{A-l} \text{ if } l+1 \leq A \leq 2l, \\ \theta_{A-2l} \text{ if } 2l+1 \leq A \leq 2l+n, \end{cases} \tag{3}$$

we can write α in the following way

$$\alpha = dz + \omega_{rs} q^r dq^s,$$

$$(\omega_{rs}) = \left(\begin{array}{cc|c} 0 & \mathrm{id}_l & 0 \\ -\mathrm{id}_l & 0 & 0 \\ \hline 0 & 0 & \mathrm{id}_n \end{array}\right).$$

Remark 5.2. We denote by ω^{sk} the matrix so that $(\omega_{rs})(\omega^{sk}) = (\delta_r^k)$. We have thus

$$(\omega^{rs}) = \left(\begin{array}{cc|c} 0 & -\mathrm{id}_l & 0 \\ \mathrm{id}_l & 0 & 0 \\ \hline 0 & 0 & \mathrm{id}_n \end{array}\right).$$

and $(\omega^{rs}) = -(-1)^{\tilde{r}\tilde{s}}(\omega^{sr})$.

Definition 5.3. We call the field of Reeb on $\mathbb{R}^{2l+1|n}$, the vector field $T_0 \in \mathrm{Vect}(\mathbb{R}^{2l+1|n})$ which, in the system of Darboux coordinates, one write $T_0 = \partial_z$.

We can show that the field of Reeb is the unique vector field on $\mathbb{R}^{2l+1|n}$ so that $i(T_0)\alpha = 1$ and $i(T_0)d\alpha = 0$.

Proposition 5.4. *In the system of Darboux coordinates, the elements* T_r *of* $\mathrm{Tan}(\mathbb{R}^{2l+1|n})$ *can be written as follows*

$$T_r = \begin{cases} A_r & := \partial_{x_r} + y_r \partial_z \ \ if\ \ 1 \le r \le l, \\ -B_{r-l} & := \partial_{y_{r-l}} - x_{r-l}\partial_z \ \ if\ \ l+1 \le r \le 2l, \\ \overline{D}_{r-2l} & := \partial_{\theta_{r-2l}} - \theta_{r-2l}\partial_z \ \ if\ \ 2l+1 \le r \le 2l+n. \end{cases} \tag{4}$$

Proof. If we denote by T_r the vector field $T_r = \partial_{q^r} - \langle \partial_{q^r}, \alpha \rangle \partial_z$, and since $\tilde{\alpha} = 0$, we have

$$\alpha(T_r) = \langle T_r, \alpha \rangle = \langle \partial_{q^r}, \alpha \rangle - \langle \langle \partial_{q^r}, \alpha \rangle \partial_z, \alpha \rangle = \langle \partial_{q^r}, \alpha \rangle - \langle \partial_{q^r}, \alpha \rangle = 0.$$

We can also show that any vector field X of $\mathrm{Tan}(\mathbb{R}^{2l+1|n})$ can be written as a linear combination of the vector fields T_r. It is useful to compute the vector fields T_r according to the matrix ω. One has, via 1,

$$T_r = \partial_{q^r} - \alpha_r \partial_z = \partial_{q^r} - \omega_{kr} q^k \partial_z.$$

It is sufficient to vary r in the interval $[1, 2l + n]$ to conclude.

The following formulas are immediate.

$$T_r(q^k) = \delta_r^k, \quad T_r(z) = -\omega_{kr} q^k,$$

$$[T_r, T_j] = -2\omega_{rj}\partial_z, \quad T_r(z^2) = -2z\omega_{kr} q^k. \tag{5}$$

6. Contact Vector Fields on $\mathbb{R}^{2l+1|n}$

Definition 6.1. We call a contact vector field on $\mathbb{R}^{2l+1|n}$ a vector field X that preserves the contact structure, i.e., a vector field X verifying the following condition: $[X, T] \in \mathrm{Tan}(\mathbb{R}^{2l+1|n})$ for all $T \in \mathrm{Tan}(\mathbb{R}^{2l+1|n})$.

The following proposition is known in the classical geometry [4] and in supergeometry in small dimensions, i.e., $(1|1)$ and $(1|2)$ in [7, 11, 9]. We give here its analogue in supergeometry and generalize it in dimension $(m|n)$. It is our main first result.

Proposition 6.2. *A vector field X on* $\mathbb{R}^{2l+1|n}$ *is called contact vector field if and only if there exists a superfunction f such that* $X = X_f$, *where* X_f *is given by the following formula*

$$X_f = f\partial_z - \frac{1}{2}(-1)^{\tilde{f}\tilde{T}_r}\omega^{rs}T_r(f)T_s, \tag{6}$$

We denote by $\mathcal{K}(2l+1|n)$, *the space of the all contact vector fields on* $\mathbb{R}^{2l+1|n}$.

Proof. Seeing the definition of T_r, we can say that any vector field on $\mathbb{R}^{2l+1|n}$ can be written as $X = f\partial_z + \sum_{i=1}^{2l+n} g_iT_i$. The vector field *X* is thus called contact vector field if and only if

$$\left[f\partial_z + \sum_{i=1}^{2l+n} g_iT_i, T_j\right] \in < T_1, \cdots, T_{2l+n} >, \quad \forall j \in \{1, \cdots, 2l+n\}.$$

This formula can be also written as

$$\left[f\partial_z + \sum_{i=1}^{2l+n} g_iT_i, T_j\right] = -(-1)^{\tilde{f}\tilde{T}_j}[T_j, f\partial_z] - (-1)^{(\tilde{g}_i+\tilde{T}_i)\tilde{T}_j}\sum_{i=1}^{2l+n}[T_j, g_iT_i]$$

$$= -(-1)^{\tilde{f}\tilde{T}_j}T_j(f)\partial_z + (-1)^{\tilde{T}_i\tilde{T}_j}2\sum_{i=1}^{2l+n} g_i\omega_{ji}\partial_z - (-1)^{(\tilde{g}_i+\tilde{T}_i)\tilde{T}_j}\sum_{i=1}^{2l+n}T_j(g_i)T_i.$$

This vector field *X* is in the kernel of α if and only if

$$-(-1)^{\tilde{f}\tilde{T}_j}T_j(f) - 2\sum_{i=1}^{2l+n} g_i\omega_{ij} = 0$$

for all $j \in \{1, \cdots, 2l+n\}$. This equation shows that all proposed vector fields X_f are contact vector fields. On the other hand, this equation implies also that

$$-(-1)^{\tilde{f}\tilde{T}_j}T_j(f)\omega^{jk} - 2\sum_{i=1}^{2l+n} g_i\omega_{ij}\omega^{jk} = 0, \quad \forall j \in \{1, \cdots, 2l+n\}$$

or, when we sum on *j*, we have

$$-(-1)^{\tilde{f}\tilde{T}_j}\,\omega^{jk}T_j(f) = 2\sum_{i=1}^{2l+n} g_i\omega_{ij}\omega^{jk} = 2\sum_{i=1}^{2l+n} g_i\delta_i^k .$$

We obtain directly that

$$g_k = -\frac{1}{2}(-1)^{\tilde{f}\tilde{T}_j}\,\omega^{jk}T_j(f)$$

and this allows us to conclude.

The following proposition gives, in the super case, the formula of the Lagrange bracket of the superfunctions *f* and *g*. It is the generalization of the formula given in [11, 7, 9].

Proposition 6.3. *The set* $\mathcal{K}(2l+1|n)$ *is a Lie sub superalgebra of* $\mathrm{Vect}(\mathbb{R}^{2l+1|n})$. *More explicitly, if* X_f *and* X_g *are the elements of* $\mathcal{K}(2l+1|n)$, *one writes*

$$[X_f, X_g] = X_{\{f, g\}}, \tag{7}$$

where the superfunction $\{f, g\}$ *is given by*

$$\{f, g\} := fg' - f'g - \frac{1}{2}(-1)^{\tilde{T}_r\tilde{f}}\,\omega^{rs}T_r(f)T_s(g) \tag{8}$$

and where $h' = \partial_z(h)$.

Proof. The Lie bracket $[X_f, X_g]$ of the two contact vector fields X_f and X_g is also a contact vector field. Indeed, the Lie bracket

$$[X_f, X_g] = \left[f\partial_z - \frac{1}{2}(-1)^{\tilde{T}_r\tilde{f}}\,\omega^{rs}T_r(f)T_s,\ g\partial_z - \frac{1}{2}(-1)^{\tilde{T}_k\tilde{g}}\,\omega^{kl}T_k(g)T_l\right]$$

is written as

$$[f\partial_z, g\partial_z] - \frac{1}{2}(-1)^{\tilde{T}_k\tilde{g}}\,\omega^{kl}[f\partial_z, T_k(g)T_l] - \frac{1}{2}(-1)^{\tilde{T}_r\tilde{f}}\,\omega^{rs}[T_r(g)T_s, g\partial_z]$$

$$+\frac{1}{4}\omega^{rs}\omega^{kl}[T_r(f)T_s, T_k(g)T_l].$$

The sum of the first three Lie brackets equals to

$$(fg'-f'g)\partial_z + \frac{1}{2}(-1)^{\tilde{g}(\tilde{T}_k+\tilde{f})}\omega^{kl}T_k(g)T_l(f)\partial_z - \frac{1}{2}(-1)^{\tilde{f}\tilde{T}_r}\omega^{rs}T_r(f)T_s(g)\partial_z$$

$$-\frac{1}{2}(-1)^{\tilde{g}\tilde{T}_k}\omega^{kl}fT_k(g')T_s + \frac{1}{2}(-1)^{\tilde{f}\tilde{T}_r+\tilde{f}\tilde{g}}\omega^{rs}gT_r(f')T_s$$

and the fourth Lie bracket equals to

$$\frac{1}{4}(-1)^{\tilde{f}\tilde{T}_r+\tilde{g}\tilde{T}_k}\omega^{rs}\omega^{kl}(T_r(f)T_sT_k(g)T_l - (-1)^{\tilde{f}\tilde{g}}T_k(g)T_lT_r(f)T_s)$$

$$-\frac{1}{2}(-1)^{(\tilde{T}_r+\tilde{f})\tilde{g}}\omega^{kr}T_k(g)T_r(f)\partial_z.$$

Since the Lie bracket of two contact vector fields is also a contact vector field and since $X_{\{f,g\}}$ is written, via the formula (6), by

$$(\{f,\ g\})\partial_z - \frac{1}{2}(-1)^{(\tilde{f}+\tilde{g})\tilde{T}_r}\omega^{rs}T_r(\{f,\ g\})T_s,$$

then we can see that the sum of the coefficients of ∂_z gives the formula of Lagrange bracket. One has thus

$$\{f,\ g\} = fg'-f'g - (-1)^{\tilde{f}\tilde{T}_r}\frac{1}{2}\omega^{rs}T_r(f)T_s(g).$$

Via the Lagrange formula (7), the Lie bracket of contact vector fields, which defines a Lie superalgebra structure on $\mathcal{K}(2l+1|n)$, induces a Lie superalgebra structure on the superspace $C^\infty(\mathbb{R}^{2l+1|n})$ by the bilinear law given by (8).

The following remark is very important:

Remark 6.4. The Lagrange bracket of superfunctions *f* and *g* of degree to most equal two is always a superfunction of degree to most equal two.

This remark allows us to define the Lie superalgebra constituted by the contact vector fields X_f which the associated superfunctions *f* are of degrees to most equal two in z, x_i, y_i and θ_i variables. We denote this Lie superalgebra temporarily by $\mathfrak{g} \subset \mathcal{K}(2l+1|n)$.

7. Matrix Realization of $\mathfrak{spo}(2l+2|n)$

In this section, we embed a Lie sub-superalgebra $\mathfrak{spo}(2l+2|n)$ of $\mathfrak{gl}(2l+2|n)$ in the Lie superalgebra $\mathrm{Vect}(\mathbb{R}^{2l+1|n})$. We use the method used in [6] and we show that the Lie superalgebra obtained is exactly isomorphic to $\mathfrak{g}$.

We consider a matrix G defined by $G = \begin{pmatrix} J & 0 \\ 0 & id_n \end{pmatrix}$ such that $J = \begin{pmatrix} 0 & -id_{l+1} \\ id_{l+1} & 0 \end{pmatrix}$. We define on $\mathbb{R}^{2l+2|n}$ the following superskewsymmetric form ω associated to the matrix G as

$$\omega : \mathbb{R}^{2l+2|n} \times \mathbb{R}^{2l+2|n} \to \mathbb{R} : (U, V) \to V^t G U, \tag{9}$$

where A^t is the usual transpose of the matrix A.

Definition 7.1. We define a Lie superalgebra $\mathfrak{spo}(2l+2|n)$ constituted by the matrices A of $\mathfrak{gl}(2l+2|n)$ which preserve the form ω, i.e., such that

$$\omega(AU, V) + (-1)^{\tilde{A}\tilde{U}}\omega(U, AV) = 0, \quad \forall U, V \in \mathbb{R}^{2l+2|n}. \tag{10}$$

Our second main result is the following:

Theorem 7.2. *The Lie superalegbra* $\mathfrak{spo}(2l+2|n)$ *is the space of the matrices* $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ *that the blocks* A_1, A_2, A_3 *and* A_4 *satisfy the following conditions*

(1) $A_1^t J + JA_1 = 0$, i.e., $A_1 \in \mathfrak{sp}(2l+2)$,

(2) $A_4^t + A_4 = 0$, i.e., $A_4 \in \mathfrak{o}(n)$,

(3) $A_3 = -A_2^t J$.

Proof. We consider the following matrices

$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathfrak{gl}(2l+2|n)$; where $A_1 \in \mathfrak{gl}(2l+2)$, $A_2 \in \mathbb{R}^n_{2l+2}$, $A_3 \in \mathbb{R}^{2l+2}_n$, $A_4 \in \mathbb{R}^n_n$. For all vector fields $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ of $\mathbb{R}^{2l+2|n}$, we compute the matrices A of $\mathfrak{gl}(2l+2|n)$ which satisfy (10). First, we can see that the first term $\omega(AU, V)$ of (10) equals to

$$V_1^t JA_1U_1 + V_2^t A_3U_1 + V_1^t JA_2U_2 + V_2^t A_4U_2$$

and the second term $(-1)^{\tilde{A}\tilde{U}}\,\omega(U, AV)$ equals to

$$V_1^t A_1^t JU_1 + V_2^t A_2^t JU_1 - V_1^t A_3^t U_2 + V_2^t A_4^t U_2.$$

It is also easy to see that the formula (10) equals to

$$V_1^t JA_1U_1 + V_2^t A_3U_1 + V_1^t JA_2U_2 + V_2^t A_4U_2 + V_1^t A_1^t JU_1$$
$$+ V_2^t A_2^t JU_1 - V_1^t A_3^t U_2 + V_2^t A_4^t U_2 = 0, \ \forall U, V \in \mathbb{R}^{2l+2|n}. \quad (11)$$

In particular, if $U_2 = 0, V_2 = 0,$ then the equation (11) equals to

$$V_1^t (JA_1 + A_1^t J)U_1 = 0, \text{ i.e., } JA_1 + A_1^t J = 0.$$

This last condition means that the blocks A_1 are symplectic matrices.

If we set $U_1 = 0$ and $V_1 = 0,$ then the equation (11) becomes

$$V_2^t (A_4^t + A_4)U_2 = 0, \text{ i.e., } A_4^t + A_4 = 0.$$

This condition means that the block A_4 is an orthogonal matrix.

Finally, if $U_2 = 0$ and $V_1 = 0,$ then the equation (11) equals to

$$V_2^t (A_3 + A_2^t J)U_1 = 0 \text{ i.e., } A_3 + A_2^t J = 0.$$

In the following, we describe a basis of $\mathfrak{spo}(2l+2|n)$ If we denote by $a_{i,\,j}$ the number $a \in \mathbb{R}$ situated on the ith line and on the jth column, we can see that this basis is constituted by the following three types of matrices:

The first type of matrices of the basis of $\mathfrak{spo}(2l+2|n)$. is associated to the symplectic algebra $\mathfrak{sp}(2l+2)$ and is given by following family of matrices:

$$\left(\begin{array}{c|c|c} 1_{i,j} & 0 & 0 \\ \hline & \quad\quad\quad 0 & \\ 0 & -1_{(l+1+j),(l+1+i)} & 0 \\ & 0 \quad\quad\quad & \\ \hline 0 & 0 & 0_{n,n} \end{array}\right); \left(\begin{array}{c|c|c} & 0 \quad 1_{i,(l+1+j)} & \\ 0 & 1_{j,(l+1+i)} & 0 \\ & \quad\quad 0 & \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0_{n,n} \end{array}\right),$$

$$\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 \quad 1_{(l+1+i),j} & & \\ 1_{(l+1+j),i} & 0 & 0 \\ \quad\quad 0 & & \\ \hline 0 & 0 & 0_{n,n} \end{array}\right) \quad 1 \le i, j \le l. \tag{12}$$

The second type of matrices is given by the following family of matrices:

$$\left(\begin{array}{c|c} 0 & 1_{(i,(2l+2+j))} \\ \hline -1_{((2l+2+j),i-(l+1))} & 0 \end{array}\right) \quad \text{if} \quad l+1 \le i \le 2l+2, \quad 1 \le j \le n \tag{13}$$

$$\text{and} \quad \left(\begin{array}{c|c} 0 & 1_{(i,(2l+2+j))} \\ \hline 1_{((2l+2+j),(l+1+i))} & 0 \end{array}\right) \quad \text{if} \quad 1 \le i \le l+1, \quad 1 \le j \le n.$$

And the third type is associated to the orthogonal algebra $\mathfrak{o}(n)$ and is given by:

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 0 & 1_{((2l+2+i),(2l+2+j))} \\ -1_{((2l+2+j),(2l+2+i))} & 0 \end{matrix} \end{array}\right). \tag{14}$$

Our third main result is given by the following theorem:

Theorem 7.3. *The Lie superalgebra* $\mathfrak{g}$ *made in evidence at the end of the Section 6 and whose superfunctions f are degrees to most equal two is isomorphic to the Lie superalgebra* $\mathfrak{spo}(2l+2|n)$.

Proof. Because of $Id \notin \mathfrak{spo}(2l+2|n)$, we can define the injective homomorphism

$$\iota : \mathfrak{spo}(2l+2|n) \to \mathfrak{pgl}(2l+2|n) : A \mapsto [A]. \tag{15}$$

Now, the Lie superalgebra $\mathfrak{pgl}(2l+2|n)$ can be embedded into the Lie superalgebra of vector fields on $\mathbb{R}^{2l+1|n}$ thanks to the projective embedding defined in [6] in the following way:

$$\left[\begin{pmatrix} 0 & \xi \\ v & B \end{pmatrix}\right] \mapsto -\sum_{i=1}^{2l+1+n} v^i \partial_{t^i} - \sum_{i,j=1}^{2l+1+n} (-1)^{\tilde{j}(\tilde{i}+\tilde{j})} B^i_j t^j \partial_{t^i}$$
$$+\sum_{i,j=1}^{2l+1+n} (-1)^{\tilde{j}} \xi_j t^j t^i \partial_{t^i}, \tag{16}$$

where $v \in \mathbb{R}^{2l+1|n}, \xi \in (\mathbb{R}^{2l+1|n})^*, B \in \mathfrak{gl}(2l+1|n)$ and the coordinates $t^1, t^2, \cdots,$ t^{2l+1+n} corresponds, respectively, $x_1, \cdots, x_l, z, y_1, \cdots, y_l, \theta_1, \cdots, \theta_n$.

Composing ι with the projective embedding, we can embed $\mathfrak{spo}(2l+2|n)$ into $\mathrm{Vect}(\mathbb{R}^{2l+1|n})$. If we compute this embedding on the generators of $\mathfrak{spo}(2l+2|n)$ written above, we obtain via (6), the contact projective vector fields X_f for a certain given $f \in C^\infty(\mathbb{R}^{2l+1|n})$. In the following, we study explicitly the three types of matrices described above. For the first matrix of (12), i.e., when $i, j \in [1, l]$, we obtain the following contact projective vector fields.

(1) If $i = j = 1$, we have

$$x_i\partial_{x_i} + y_i\partial_{y_i} + \theta_i\partial_{\theta_i} + 2z\partial_z, \text{ i.e., } 2X_z.$$

(2) If $i = 1$ and $j \neq 1$, then we have

$$x_{j-1}(x_i\partial_{x_i} + y_i\partial_{y_i} + z\partial_z + \theta_i\partial_{\theta_i}) + z\partial_{y_{j-1}}, \text{ i.e., } 2X_{x_{j-1}z}.$$

(3) If $i \neq 1$ and $j = 1$, then we obtain

$$-\partial_{x_{i-1}} + y_{i-1}\partial_z, \text{ i.e., } 2X_{y_{i-1}}.$$

(4) If $i \neq 1$ and $j \neq 1$, then

$$y_{i-1}\partial_{y_{j-1}} - x_{j-1}\partial_{x_{i-1}}, \text{ i.e., } 2X_{x_{i-1}y_{i-1}}.$$

For the second matrix of (12), we obtain the following contact projective vector fields

(1) If $i = j = 1$, one has

$$z(z\partial_z + x_i\partial_{x_i} + y_i\partial_{y_i} + \theta_j\partial_{\theta_j}), \text{ i.e., } X_{z^2}.$$

(2) If $i = j$ and $j \neq 1$, we have

$$-y_{i-1}\partial_{x_{i-1}}, \text{ i.e., } X_{y_{i-1}^2}.$$

(3) If $i \neq j$ and $i = 1$, one has

$$y_{j-1}(x_i\partial_{x_i} + y_i\partial_{y_i} + z\partial_z + \theta_j\partial_{\theta_j}) - z\partial_{x_{j-1}}, \text{ i.e., } 2X_{y_{j-1}z}.$$

(4) If $i \neq j$ and $i \neq 1$, one has

$$-(y_{j-1}\partial_{x_{i-1}} + y_{i-1}\partial_{x_{j-1}}), \text{ i.e., } 2X_{y_{i-1}y_{j-1}}.$$

For the third matrix of (12), we obtain the following contact projective vector fields

(1) If $i = j = 1$, we obtain

$$-\partial_z, \text{ i.e., } -X_1.$$

(2) If $i = j$ and $j \neq 1$, we have

$$-x_{i-1}\partial_{y_{i-1}}, \text{ i.e., } -X_{x_{i-1}^2}.$$

(3) If $i \neq j$ and $i = 1$, we write

$$-(\partial_{y_{j-1}} + x_{j-1}\partial_z), \text{ i.e., } -2X_{x_{j-1}}.$$

(4) If $i \neq j$ and $i \neq 1$, then we have

$$-(x_{j-1}\partial_{y_{i-1}} + x_{i-1}\partial_{y_{j-1}}), \text{ i.e., } -2X_{x_{j-1}x_{i-1}}.$$

Now, we study explicitly any matrix of (13), i.e., when $l + 1 \leq i \leq 2l + 2$ and $1 \leq j \leq n$. For the first matrix of (13), we obtain the following contact projective vector fields

(1) If $i = l + 2$, then we have

$$\theta_j\partial_z + \partial_{\theta_j}, \text{ i.e., } 2X_{\theta_j}$$

(2) and if $i \neq l + 2$, we obtain

$$\theta_j\partial_{y_{i-l-2}} + x_{i-l-2}\partial_{\theta_j}, \text{ i.e., } 2X_{x_{i-l-2}\theta_j}.$$

For the second matrix of (13), we have the following contact projective vector fields

(1) If $i = 1$, one has

$$-\theta_j(x_i\partial_{x_i} + y_i\partial_{y_i} + z\partial_z + \theta_j\partial_{\theta_j}) - z\partial_{\theta_j}, \text{ i.e., } -2X_{z\theta_j}$$

(2) and if $i \neq 1$, then we obtain

$$\theta_j\partial_{x_{i-1}} - y_{i-1}\partial_{\theta_j}, \text{ i.e., } -2X_{y_{i-1}\theta_j}.$$

Finally, for the matrix (14), i.e., when $2l + 2 \leq i,\ j \leq 2l + 2 + n$, we obtain the contact projective vector field

$$\theta_{i-1}\partial_{\theta_{j-1}} - \theta_{j-1}\partial_{\theta_{i-1}}, \; i < j, \text{ i.e., } 2X_{\theta_{i-1}\theta_{j-1}}.$$

The Lie superalgebra $\mathfrak{g}$ is isomorphic to $\mathfrak{spo}(2l+2|n)$ thanks to the identification of the generators of these two Lie superalgebras.

Remark 7.4. The Lie superalgebra $\mathfrak{spo}(2l+2|n)$ is into the intersection of the Lie superalgebra of contact vector fields $\mathcal{K}(2l+1|n)$ and the Lie superalgebra of projective vector fields $\mathfrak{pgl}(2l+2|n)$. Thus, as in [4], the elements X_f of $\mathfrak{spo}(2l+2|n)$ are called contact projective vector fields.

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