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ON SOME EQUATIONS OF THE LAPLACIAN TYPE ARISING FROM DISCRETE RANDOM WALK PROBLEMS WITH POSITION DEPENDENT JUMP PROBABILITIES IN 4D

CLEMENT BOATENG AMPADU

31 Carrolton Road
Boston, MA 02132-6303
USA
e-mail: drampadu@hotmail.com

Abstract

Let A be a bounded region in 4D, and let ∂A be its surface boundary which we assume to be absorbing. Enclose A and its boundary with a 4-parallelotope all of whose faces are possibly square, and let the sides be given by $x = a_i, a_{i+1}$ for $i = 1$; $y = a_i, a_{i+1}$ for $i = 3$; $z = a_i, a_{i+1}$ for $i = 5$; $m = a_i, a_{i+1}$ for $i = 7$. Let δ be the step length in the random walk, and assume that the intervals $[a_i, a_{i+1}]$ for $i = 1, 3, 5, 7$ can be subdivided into the set of points

$$x_{k_1} = a_1 + \delta k_1,\ x_{n_1} = a_2,\ 0 \le k_1 \le n_1,$$

$$y_{k_2} = a_3 + \delta k_2,\ y_{n_2} = a_4,\ 0 \le k_2 \le n_2,$$

$$z_{k_3} = a_5 + \delta k_3,\ z_{n_3} = a_6,\ 0 \le k_3 \le n_3,$$

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$$m_{k_4} = a_7 + \delta k_4,\ m_{n_4} = a_8,\ 0 \le k_4 \le n_4.$$

We say $(x_{k_1}, y_{k_2}, z_{k_3}, m_{k_4})$ is an interior point of A, if it does not lie on ∂A. If one of the neighboring points lies on ∂A or is exterior to A, we call it a boundary point. The points in the 4-parallelotope that are neither interior nor boundary points shall not be considered. In this paper, working primarily in 4D, we study the following problems:

(a) What is the probability that a particle starting at some point in the region reaches a certain point on the boundary and is absorbed before it reaches and is absorbed by the remaining portion of the boundary?

(b) What is the probability that a particle starting at some point in the region reaches a fixed interior point before it is absorbed by the boundary?

(c) What is the mean or expected time it takes for a particle starting at some point in the interior until it is absorbed at the boundary?

1. Introduction

Remark 1.1. Although we speak of the mean or expected time in the third random walk problem of the abstract, this problem is actually time-independent since the possible times until absorption for each interior point are averaged out, so the time-dependence is not explicit.

In this paper, we assume the steps taken by the particle are uncorrelated, meaning that each step taken is completely independent of the previous steps taken, as such the motion is Brownian. Furthermore, we do not concern ourselves with the number of steps required for the particle to reach the fixed boundary or interior point, that is, the investigation of the problems will take place independent of time.

Notation 1.2. In this paper we will use the following:

(a) p_1 for the probability a particle located at the point (x, y, z, m) is moving in the negative direction of the x-axis.

(b) q_1 for the probability a particle located at the point (x, y, z, m) is moving in the positive direction of the x-axis.

(c) p_2 for the probability a particle located at the point (x, y, z, m) is moving in the negative direction of the y-axis.

(d) q_2 for the probability a particle located at the point (x, y, z, m) is moving in the positive direction of the y-axis.

(e) p_3 for the probability a particle located at the point (x, y, z, m) is moving in the negative direction of the z-axis.

(f) q_3 for the probability a particle located at the point (x, y, z, m) is moving in the positive direction of the z-axis.

(g) p_4 for the probability a particle located at the point (x, y, z, m) is moving in the negative direction of the m-axis.

(h) q_4 for the probability a particle located at the point (x, y, z, m) is moving in the positive direction of the m-axis.

Observe that in 4D the conditional probability of moving in any of the axial directions is $\frac{1}{8}$. Now we introduce the following:

$$p_i(x, y, z, m) = \frac{1}{8}[a_i(x, y, z, m) + b_i(x, y, z, m)\delta], \text{ for } i = 1, 2, 3, 4, \quad (1.1)$$

$$q_i(x, y, z, m) = \frac{1}{8}[a_i(x, y, z, m) - b_i(x, y, z, m)\delta], \text{ for } i = 1, 2, 3, 4. \quad (1.2)$$

In the above a_i and b_i are certain smooth functions depending on the position of the particle and choosen such that $0 < \sum_{i=1}^{4}(p_i + q_i) \le 1.$ Notice that $\sum_{i=1}^{4}(p_i + q_i) = \frac{1}{4}\sum_{i=1}^{4} a_i$. It follows that if $\frac{1}{4}\sum_{i=1}^{4} a_i < 1,$ then there is a nonzero probability

$$1 - \sum_{i=1}^{4}(p_i + q_i) = 1 - \frac{1}{4}\sum_{i=1}^{4} a_i$$

that the particle rest at each step in 4D, that is, takes a step of zero length.

This paper is organized as follows. In Section 2, we obtain a generalization of Laplace's equation in 4D. In Chapter II of [1], we obtained an inhomogeneous form of Laplace's equation in 3D. The main result in Section 3, is a position-dependent partial differential equation, that can be tied to an inhomogeneous form of Laplace's equation in 4D. This inhomogeneous form of Laplace's equation in 4D can be viewed as natural extension of the result obtained in Chapter II [1]. As a corollary, the adjoint of the position-dependent partial differential equation is obtained. The main result in Section 4, is a position-dependent partial differential equation, which under certain conditions reduces to a four-dimensional generalization of Theorem II.5.1 [1] which arises from a position-independent analogue of the third random walk problem.

2. Result for the First Random Walk Problem

2.1. The difference equation

Recall that the first random walk problem asks for the probability that the particle starting at the interior point of A reaches the fixed boundary point $(x_{w1}, y_{w2}, z_{w3}, m_{w4})$ before it reaches any other boundary point. Let $v(x, y, z, m)$ be the probability that the particle starts at the interior point (x, y, z, m) and reaches the boundary point $(x_{w1}, y_{w2}, z_{w3}, m_{w4})$, then we see the difference equation model for this problem is given by

$$\begin{aligned} v(x, y, z, m) = {} & [1 - \sum_{i=1}^{4}(p_i + q_i)]v(x, y, z, m) \\ & + p_1 v(x + \delta, y, z, m) + q_1 v(x - \delta, y, z, m) \\ & + p_2 v(x, y + \delta, z, m) + q_2 v(x, y - \delta, z, m) \\ & + p_3 v(x, y, z + \delta, m) + q_3 v(x, y, z - \delta, m) \\ & + p_4 v(x, y, z, m + \delta) + q_4 v(x, y, z, m - \delta). \end{aligned} \tag{2.1}$$

2.2. Taylor expansions in the right hand-side of (2.1)

Expanding the terms $v(x \pm \delta, y, z, m)$, $v(x, y \pm \delta, z, m)$, $v(x, y, z \pm \delta, m)$, $v(x, y, z, m \pm \delta)$ using Taylor's formula gives

$$v(x \pm \delta, y, z, m) = v(x, y, z, m) \pm \delta v_x(x, y, z, m) + \frac{1}{2}\delta^2 v_{xx}(x, y, z, m) + O(\delta^3), \tag{2.2}$$

$$v(x, y \pm \delta, z, m) = v(x, y, z, m) \pm \delta v_y(x, y, z, m) + \frac{1}{2}\delta^2 v_{yy}(x, y, z, m) + O(\delta^3), \tag{2.3}$$

$$v(x, y, z \pm \delta, m) = v(x, y, z, m) \pm \delta v_z(x, y, z, m) + \frac{1}{2}\delta^2 v_{zz}(x, y, z, m) + O(\delta^3), \tag{2.4}$$

$$v(x, y, z, m \pm \delta) = v(x, y, z, m) \pm \delta v_m(x, y, z, m) + \frac{1}{2}\delta^2 v_{mm}(x, y, z, m) + O(\delta^3). \tag{2.5}$$

2.3. A useful lemma

Lemma 2.1. *We have the following*

(a) $\sum_{i=1}^{4}(p_i + q_i) = \frac{1}{4}\sum_{i=1}^{4} a_i$,

(b) $p_i - q_i = \frac{1}{4} b_i \delta$, *for* $i = 1, 2, 3, 4$,

(c) $p_i + q_i = \frac{1}{4} a_i$, *for* $i = 1, 2, 3, 4$.

Proof. It follows directly from Equations (1.1) and (1.2).

2.4. The main theorem

Now our main result for the first random walk problem is as follows:

Theorem 2.2. *With the jump probabilities given as in* (1.1) *and* (1.2), *the limiting partial differential equation arising from* (2.1) *is given by*

$$0 = a_1(x, y, z, m)v_{xx} + a_2(x, y, z, m)v_{yy}$$

$$+a_3(x, y, z, m)v_{zz} + a_4(x, y, z, m)v_{mm}$$

$$+2b_1(x, y, z, m)v_x + 2b_2(x, y, z, m)v_y$$

$$+2b_3(x, y, z, m)v_z + 2b_4(x, y, z, m)v_m$$

and the boundary condition is given by $v(x, y, z, m) = 0$, $(x, y, z, m) \in \partial A$, $(x, y, z, m) \neq (\hat{x}, \hat{y}, \hat{z}, \hat{m})$, $\iint_{\partial A} v(x, y, z, m)dQ = 1$.

Proof. Substituting (2.2)-(2.5) into (2.1) and simplifying gives

$$0 = (p_1 - q_1)\delta v_x + (p_2 - q_2)\delta v_y + (p_3 - q_3)\delta v_z + (p_4 - q_4)\delta v_m$$

$$+ \frac{1}{2}(p_1 + q_1)\delta^2 v_{xx} + \frac{1}{2}(p_2 + q_2)\delta^2 v_{yy}$$

$$+ \frac{1}{2}(p_3 + q_3)\delta^2 v_{zz} + \frac{1}{2}(p_4 + q_4)\delta^2 v_{mm}$$

$$+ \left[\sum_{i=1}^{4}(p_i + q_i)\right]O(\delta^3). \tag{2.6}$$

Now using Lemma 2.1 in (2.6) and simplifying gives

$$0 = \frac{1}{4}b_1\delta^2 v_x + \frac{1}{4}b_2\delta^2 v_y + \frac{1}{4}b_3\delta^2 v_z + \frac{1}{4}b_4\delta^2 v_m$$

$$+ \frac{1}{8}a_1\delta^2 v_{xx} + \frac{1}{8}a_2\delta^2 v_{yy} + \frac{1}{8}a_3\delta^2 v_{zz} + \frac{1}{8}a_4\delta^2 v_{mm}$$

$$+ \left[\frac{1}{4}\sum_{i=1}^{4}a_i\right]O(\delta^3). \tag{2.7}$$

Now dividing by δ^2 in (2.7) and letting $\delta \to 0$ gives

$$0 = \frac{1}{4}b_1 v_x + \frac{1}{4}b_2 v_y + \frac{1}{4}b_3 v_z + \frac{1}{4}b_4 v_m$$

$$+\frac{1}{8}a_1v_{xx} + \frac{1}{8}a_2v_{yy} + \frac{1}{8}a_3v_{zz} + \frac{1}{8}a_4v_{mm}. \tag{2.8}$$

So the desired result is obtained by multiplying (2.8) by 8. For the boundary condition, since $v(x, y, z, m)$ is the probability that a particle starts at the interior point (x, y, z, m) and reaches some fixed boundary point, if (x, y, z, m) is a boundary point and is taken to be the fixed boundary point in question, then $v(x, y, z, m) = 1$, since the particle is already there to begin with, however, if (x, y, z, m) is a boundary point different from the fixed boundary point, then $v(x, y, z, m) = 0$, since the boundary is absorbing, and thus the particle cannot reach the fixed boundary point from there. Moreover, if we assume that the area Q is defined on the boundary of the bounded region A, and that as $\delta \to 0$, the fixed boundary point tends to $(\hat{x}, \hat{y}, \hat{z}, \hat{m})$ on the boundary of the bounded region A, then it follows that

$$v(x, y, z, m) = 0, (x, y, z, m) \in \partial A, (x, y, z, m) \neq (\hat{x}, \hat{y}, \hat{z}, \hat{m})$$

and since $v(x, y, z, m,)$ is a probability density, then normalizing implies

$$\iint_{\partial A} v(x, y, z, m,)dQ = 1.$$

Remark 2.3. If in Theorem 2.2, we put

$$a_i(x, y, z, m) = 1 \text{ for } i = 1, 2, 3, 4$$

and

$$b_i(x, y, z, m) = 0 \text{ for } i = 1, 2, 3, 4,$$

then we obtain Laplace's equation in 4D. In particular, under this condition, we obtain the limiting partial differential equation arising from the 4-dimensional extension of the first random problem considered in Chapter II [1].

3. Result for the Second Random Walk Problem

3.1. The difference equation

Recall the second random walk problem asks for the probability that a particle starting at an interior point (x, y, z, m) in the region A reaches a fixed interior point $(\xi_1, \xi_2, \eta_1, \eta_2)$ (say) before it reaches a boundary point and is absorbed. The region A is subdivided as in the first problem, and interior and boundary points are defined as before with step length equal to δ. Since the problem is time independent, and the particle does not stop its motion once it reaches $(\xi_1, \xi_2, \eta_1, \eta_2)$ for the first time, it is possible for the particle to pass through the point $(\xi_1, \xi_2, \eta_1, \eta_2)$ more than once before it reaches and is absorbed at the boundary. Consequently, if the particle begins its motion at $(\xi_1, \xi_2, \eta_1, \eta_2)$ it has unit probability of reaching $(\xi_1, \xi_2, \eta_1, \eta_2)$ since it is already there to begin with. However, it can also move to one of its four neighboring points, and reach $(\xi_1, \xi_2, \eta_1, \eta_2)$ from there if the neighbor is not a boundary point. Therefore, if we introduce a function $w(x, y, z, m)$ that characterizes the prospects of a particle reaching $(\xi_1, \xi_2, \eta_1, \eta_2)$ from the starting point (x, y, z, m), we cannot consider $w(x, y, z, m)$ to be a probability distribution since it may assume values exceeding unity. Let $w(x, y, z, m)$ be the expectation of reaching $(\xi_1, \xi_2, \eta_1, \eta_2)$ from (x, y, z, m) before it is absorbed at the boundary. If $(x, y, z, m) \neq (\xi_1, \xi_2, \eta_1, \eta_2)$, then $w(x, y, z, m)$ satisfies the following:

$$
\begin{aligned}
w(x, y, z, m) = \left[1 - \sum_{i=1}^{4}(p_i + q_i)\right] & w(x, y, z, m) \\
& + p_1 w(x + \delta, y, z, m) + q_1 w(x - \delta, y, z, m) \\
& + p_2 w(x, y + \delta, z, m) + q_2 w(x, y - \delta, z, m) \\
& + p_3 w(x, y, z + \delta, m) + q_3 w(x, y, z - \delta, m)
\end{aligned}
$$

$$+p_4 w(x,\ y,\ z,\ m+\delta) + q_4 w(x,\ y,\ z,\ m-\delta). \tag{3.1}$$

If $(x, y, z, m) = (\xi_1, \xi_2, \eta_1, \eta_2)$, then $w(x, y, z, m)$ satisfies the following:

$$\begin{aligned} w(x,\ y,\ z,\ m) = 1 + &\left[1 - \sum_{i=1}^{4}(p_i + q_i)\right] w(x,\ y,\ z,\ m) \\ &+p_1 w(x+\delta,\ y,\ z,\ m) + q_1 w(x-\delta,\ y,\ z,\ m) \\ &+p_2 w(x,\ y+\delta,\ z,\ m) + q_2 w(x,\ y-\delta,\ z,\ m) \\ &+p_3 w(x,\ y,\ z+\delta,\ m) + q_3 w(x,\ y,\ z-\delta,\ m) \\ &+p_4 w(x,\ y,\ z,\ m+\delta) + q_4 w(x,\ y,\ z,\ m-\delta). \end{aligned} \tag{3.2}$$

3.2. Some Taylor expansions

Expanding the terms $p_1(x+\delta,\ y,\ z,\ m)$, $q_1(x-\delta,\ y,\ z,\ m)$, $p_2(x,\ y+\delta,\ z,\ m)$, $q_2(x,\ y-\delta,\ z,\ m)$, $p_3(x,\ y,\ z+\delta,\ m)$, $q_3(x,\ y,\ z-\delta,\ m)$, $p_4(x, y, z,\ m+\delta)$, $q_4(x,\ y,\ z,\ m-\delta)$ using Taylor's formula gives

$$\begin{aligned} p_1(x+\delta,\ y,\ z,\ m) = &\ p_1(x,\ y,\ z,\ m) + \delta(p_1)_x(x,\ y,\ z,\ m) \\ &+ \frac{1}{2}\delta^2(p_1)_{xx}(x,\ y,\ z,\ m) + O(\delta^3), \end{aligned} \tag{3.3}$$

$$\begin{aligned} q_1(x-\delta,\ y,\ z,\ m) = &\ q_1(x,\ y,\ z,\ m) - \delta(q_1)_x(x,\ y,\ z,\ m) \\ &+ \frac{1}{2}\delta^2(q_1)_{xx}(x,\ y,\ z,\ m) + O(\delta^3), \end{aligned} \tag{3.4}$$

$$\begin{aligned} p_2(x,\ y+\delta,\ z,\ m) = &\ p_2(x,\ y,\ z,\ m) + \delta(p_2)_y(x,\ y,\ z,\ m) \\ &+ \frac{1}{2}\delta^2(p_2)_{yy}(x,\ y,\ z,\ m) + O(\delta^3), \end{aligned} \tag{3.5}$$

$$\begin{aligned} q_2(x,\ y-\delta,\ z,\ m) = &\ q_2(x,\ y,\ z,\ m) - \delta(q_2)_y(x,\ y,\ z,\ m) \\ &+ \frac{1}{2}\delta^2(q_2)_{yy}(x,\ y,\ z,\ m) + O(\delta^3), \end{aligned} \tag{3.6}$$

$$p_3(x, y, z+\delta, m) = p_3(x, y, z, m) + \delta(p_3)_z(x, y, z, m)$$

$$+\frac{1}{2}\delta^2(p_3)_{zz}(x, y, z, m) + O(\delta^3), \tag{3.7}$$

$$q_3(x, y, z-\delta, m) = q_3(x, y, z, m) - \delta(q_3)_z(x, y, z, m)$$

$$+\frac{1}{2}\delta^2(q_3)_{zz}(x, y, z, m) + O(\delta^3), \tag{3.8}$$

$$p_4(x, y, z, m+\delta) = p_4(x, y, z, m) + \delta(p_4)_m(x, y, z, m)$$

$$+\frac{1}{2}\delta^2(p_4)_{mm}(x, y, z, m) + O(\delta^3), \tag{3.9}$$

$$q_4(x, y, z, m-\delta) = q_4(x, y, z, m) - \delta(q_4)_m(x, y, z, m)$$

$$+\frac{1}{2}\delta^2(q_4)_{mm}(x, y, z, m) + O(\delta^3). \tag{3.10}$$

On the other hand expanding the terms $w(x \pm \delta, y, z, m)$, $w(x, y \pm \delta, z, m)$, $w(x, y, z \pm \delta, m)$, $w(x, y, z, m \pm \delta)$ using Taylor's formula gives

$$w(x \pm \delta, y, z, m) = w(x, y, z, m) \pm \delta w_x(x, y, z, m)$$

$$+\frac{1}{2}\delta^2 w_{xx}(x, y, z, m) + O(\delta^3), \tag{3.11}$$

$$w(x, y \pm \delta, z, m) = w(x, y, z, m) \pm \delta w_y(x, y, z, m)$$

$$+\frac{1}{2}\delta^2 w_{yy}(x, y, z, m) + O(\delta^3), \tag{3.12}$$

$$w(x, y, z \pm \delta, m) = w(x, y, z, m) \pm \delta w_z(x, y, z, m)$$

$$+\frac{1}{2}\delta^2 w_{zz}(x, y, z, m) + O(\delta^3), \tag{3.13}$$

$$w(x, y, z, m \pm \delta) = w(x, y, z, m) \pm \delta w_m(x, y, z, m)$$

$$+\frac{1}{2}\delta^2 w_{mm}(x, y, z, m) + O(\delta^3). \tag{3.14}$$

From Lemma 2.1(b) and Lemma 2.1(c), define $(p_i + q_i)(x, y, z, m) := \frac{1}{4}$ $a_i(x, y, z, m)$ and $(p_i - q_i)(x, y, z, m) := \frac{1}{4}\delta b_i(x, y, z, m)$, then we have the following for each partial derivative in $J = \{x, xx, y, yy, z, zz, m, mm\}$ and each $i = 1, 2, 3, 4$

$$(p_i + q_i)_{j \in J}(x, y, z, m) = \frac{1}{4}(a_i)_{j \in J}(x, y, z, m), \tag{3.15}$$

$$(p_i - q_i)_{j \in J}(x, y, z, m) := \frac{1}{4}\delta(b_i)_{j \in J}(x, y, z, m). \tag{3.16}$$

Finally observe we have the following:

$$(a_1 w)_{xx} = (a_1)_{xx} w + 2(a_1)_x w_x + a_1 w_{xx}, \tag{3.17}$$

$$(a_2 w)_{yy} = (a_2)_{yy} w + 2(a_2)_y w_x + a_2 w_{yy}, \tag{3.18}$$

$$(a_3 w)_{zz} = (a_3)_{zz} w + 2(a_3)_z w_z + a_3 w_{zz}, \tag{3.19}$$

$$(a_4 w)_{mm} = (a_4)_{mm} w + 2(a_4)_m w_m + a_4 w_{mm}, \tag{3.20}$$

$$2(b_1 w)_x = 2(b_1)_x w + 2b_1 w_x, \tag{3.21}$$

$$2(b_2 w)_y = 2(b_2)_y w + 2b_2 w_y, \tag{3.22}$$

$$2(b_3 w)_z = 2(b_3)_z w + 2b_3 w_z, \tag{3.23}$$

$$2(b_4 w)_m = 2(b_4)_m w + 2b_4 w_m. \tag{3.24}$$

3.3. The main theorem

Now our main result for the second random walk problem is as follows:

Theorem 3.1. *With jump probabilities defined as in* (1.1) *and* (1.2), *the limiting partial differential equation arising from* (3.1) *and* (3.2) *are as follows*:

(a) *If* $(x, y, z, m) \neq (\xi_1, \xi_2, \eta_1, \eta_2)$, *then*

$$(a_1 w)_{xx} + (a_2 w)_{yy} + (a_3 w)_{zz} + (a_4 w)_{mm}$$
$$+2[(b_1 w)_x + (b_2 w)_y + (b_3 w)_z + (b_4 w)_m]$$
$$= 0.$$

(b) *If* $(x, y, z, m) = (\xi_1, \xi_2, \eta_1, \eta_2)$, *then*

$$(a_1 w)_{xx} + (a_2 w)_{yy} + (a_3 w)_{zz} + (a_4 w)_{mm}$$
$$+2[(b_1 w)_x + (b_2 w)_y + (b_3 w)_z + (b_4 w)_m]$$
$$= -8.$$

Remark 3.2. Due to the fact that the algebra involved in the final result of the above theorem is very lengthy, we only sketch the proof below:

Proof. Case I. If $(x, y, z, m) \neq (\xi_1, \xi_2, \eta_1, \eta_2)$, then do the following:

(a) Substitute (3.3)-(3.14) into (3.1) and simplify the result to get an expression similar to (2.6). We omit the terms involving "O" from the simplified expression since they automatically vanish in the continuum limit.

(b) In the simplified expression obtained from (a) substitute (3.15)-(3.16), this will result in an expression similar to (2.7).

(c) Divide the expression in step (b) by δ^2, and let $\delta \to 0$ in the result gives an expression similar to (2.8).

(d) Multiply the expression obtained in step (c) by 8.

After the multiplication in step (d) above, we obtain the following expression:

$$0 = 2b_1 w_x + a_1 w_{xx} + 2(b_1)_x w + 2(a_1)_x w_x + (a_1)_{xx} w$$
$$+2b_2 w_y + a_2 w_{yy} + 2(b_2)_y w + 2(a_2)_y w_y + (a_2)_{yy} w$$

$$+2b_3w_z + a_3w_{zz} + 2(b_3)_z w + 2(a_3)_z w_z + (a_3)_{zz} w$$

$$+2b_4w_m + a_4w_{mm} + 2(b_4)_m w + 2(a_4)_m w_m + (a_4)_{mm} w. \tag{3.25}$$

Now using (3.17)-(3.24) in (3.25) gives part (a) of the Theorem.

Case II. If $(x, y, z, m) = (\xi_1, \xi_2, \eta_1, \eta_2)$, then do the following:

(a) Substitute (3.3)-(3.14) into (3.2) and simplify the result to get an expression similar to (2.6). We omit the terms involving "*O*" from the simplified expression since they automatically vanish in the continuum limit.

(b) In the simplified expression obtained from (a) substitute (3.15)-(3.16), this will result in an expression similar to (2.7).

(c) Divide the expression in step (b) by δ^2, and let $\delta \to 0$ in the result gives an expression similar to (2.8).

(d) Multiply the expression obtained in step (c) by 8.

After the multiplication in step (d) above, we obtain the following expression:

$$0 = 8 + 2b_1w_x + a_1w_{xx} + 2(b_1)_x w + 2(a_1)_x w_x + (a_1)_{xx} w$$

$$+2b_2w_y + a_2w_{yy} + 2(b_2)_y w + 2(a_2)_y w_y + (a_2)_{yy} w$$

$$+2b_3w_z + a_3w_{zz} + 2(b_3)_z w + 2(a_3)_z w_z + (a_3)_{zz} w$$

$$+2b_4w_m + a_4w_{mm} + 2(b_4)_m w + 2(a_4)_m w_m + (a_4)_{mm} w. \tag{3.26}$$

Now using (3.17)-(3.24) in (3.26) gives part (b) of the Theorem, and the proof is completed. ,

3.4. A corollary

By Remark III.2.7 [1], the following is immediate:

Corollary 3.3. *The adjoint of the partial differential equations in Theorem* 3.1 *are as follows*:

(a) *If* $(x, y, z, m) \neq (\xi_1, \xi_2, \eta_1, \eta_2)$, *then*

$$(a_1 w)_{xx} + (a_2 w)_{yy} + (a_3 w)_{zz} + (a_4 w)_{mm}$$
$$-2[(b_1 w)_x + (b_2 w)_y + (b_3 w)_z + (b_4 w)_m]$$
$$= 0.$$

(b) *If* $(x, y, z, m) = (\xi_1, \xi_2, \eta_1, \eta_2)$, *then*

$$(a_1 w)_{xx} + (a_2 w)_{yy} + (a_3 w)_{zz} + (a_4 w)_{mm}$$
$$-2[(b_1 w)_x + (b_2 w)_y + (b_3 w)_z + (b_4 w)_m]$$
$$= -8.$$

Remark 3.4. The above Corollary implies that if $K(x, y, z, m; \xi_1, \xi_2, \eta_1, \eta_2)$ is the Green's function associated with the second random walk problem, then by definition it satisfies

$$-\delta(x - \xi_1)\delta(y - \xi_2)\delta(z - \eta_1)\delta(m - \eta_2)$$
$$= (a_1 K)_{xx} + (a_2 K)_{yy} + (a_3 K)_{zz} + (a_4 K)_{mm}$$
$$-2[(b_1 K)_x + (b_2 K)_y + (b_3 K)_z + (b_4 K)_m].$$

3.5. A discussion

One consequence of the second random walk problem is that if the particle starts its motion at an interior point which is not the one we fix, then taking $a_i(x, y, z, m) = 1$ for $i = 1, 2, 3, 4$ and $b_i(x, y, z, m) = 0$ for $i = 1, 2, 3, 4$ in Theorem 3.1, implies we can write $w_{xx} + w_{yy} + w_{zz} + w_{mm} = O(\delta)$ provided that $\delta \to 0$.

Another consequence of the second random walk problem is that if the particle starts its motion at the fixed interior point, then taking $a_i(x, y, z, m) = 1$ for $i = 1, 2, 3, 4$ and $b_i(x, y, z, m) = 0$ for $i = 1, 2, 3, 4$ in Theorem 3.1, implies we can write $w_{xx} + w_{yy} + w_{zz} + w_{mm} = -8/\delta^2 + O(\delta)$ provided that $O(1) = 0$.

Both consequences above, imply that if the jump probabilities are position independent, then for small δ, the second random walk problem leads to an inhomogeneous form of Laplace's equation in 4D that can be viewed as natural extension of the result obtained in Chapter II [1]. Moreover, as $\delta \to 0$, $w(x, y, z, m)$ will satisfy typical Laplace equation in 4D, provided that the particle starts its motion at an interior point which is not the one we fix, and blows up if the particle starts its motion at the fixed interior point.

4. A Relationship between the First and Second Random Walk Problems

4.1. A useful lemma

Lemma 4.1. *Let* L *be the differential operator acting on* v *in the first random walk problem, and let* L^* *be the differential operator acting on* K *in Remark* 3.4. *Define these operators as*

$$L[v] = a_1(x, y, z, m)v_{xx} + a_2(x, y, z, m)v_{yy} + a_3(x, y, z, m)v_{zz}$$

$$+a_4(x, y, z, m)v_{mm} + 2b_1(x, y, z, m)v_x + 2b_2(x, y, z, m)v_y$$

$$+2b_3(x, y, z, m)v_z + 2b_4(x, y, z, m)v_m$$

and

$$L^*[K] = (a_1K)_{xx} + (a_2K)_{yy} + (a_3K)_{zz} + (a_4K)_{mm}$$

$$-(2b_1K)_x - (2b_2K)_y - (2b_3K)_z - (2b_4K)_m.$$

Then

$$vL^*[K] - KL[v] = [v(a_1K)_x - a_1Kv_x - 2b_1Kv]_x$$

$$+[v(a_2K)_y - a_2Kv_y - 2b_2Kv]_y$$

$$+[v(a_3K)_z - a_3Kv_z - 2b_3Kv]_z$$

$$+[v(a_4K)_m - a_4Kv_m - 2b_4Kv]_m.$$

Proof. First observe we have the following

$$\begin{aligned} vL^*[K] - KL[v] = {} & v(a_1K)_{xx} + v(a_2K)_{yy} + v(a_3K)_{zz} + v(a_4K)_{mm} \\ & -2v(b_1K)_x - 2v(b_2K)_y - 2v(b_3K)_z - 2v(b_4K)_m \\ & -a_1Kv_{xx} - a_2Kv_{yy} - a_3Kv_{zz} - a_4Kv_{mm} \\ & -2b_1Kv_x - 2b_2Kv_y - 2b_3Kv_z - 2b_4Kv_m. \end{aligned} \tag{4.1}$$

Now observe that we have the following:

$$\begin{aligned} & [v(a_1K)_x - a_1Kv_x - 2b_1Kv]_x \\ = {} & v(a_1)_{xx}K + 2v(a_1)_xK_x + va_1K_{xx} - a_1Kv_{xx} \\ & -2(b_1)_xKv - 2b_1K_xv - 2b_1Kv_x, \end{aligned} \tag{4.2}$$

$$v(a_1K)_{xx} = v(a_1)_{xx}K + 2v(a_1)_xK_x + va_1K_{xx}, \tag{4.3}$$

$$v(b_1K)_x = vb_1K_x + v(b_1)_xK. \tag{4.4}$$

Now using (4.3) and (4.4) in (4.2), we deduce the following:

$$\begin{aligned} & [v(a_1K)_x - a_1Kv_x - 2b_1Kv]_x \\ = {} & v(a_1K)_{xx} - 2v(b_1K)_x - ka_1v_{xx} + 2b_1Kv_x. \end{aligned} \tag{4.5}$$

Now observe that we have the following:

$$\begin{aligned} & [v(a_2K)_y - a_2Kv_y - 2b_2Kv]_y \\ = {} & v(a_2)_{yy}K + 2v(a_2)_yK_y + va_2K_{yy} - a_2Kv_{yy} \\ & -2(b_2)_yKv - 2b_2K_yv - 2b_2Kv_y, \end{aligned} \tag{4.6}$$

$$v(a_2K)_{yy} = v(a_2)_{yy}K + 2v(a_2)_yK_y + va_2K_{yy}, \tag{4.7}$$

$$v(b_2K)_y = vb_2K_y + v(b_2)_yK. \tag{4.8}$$

Now using (4.7) and (4.8) in (4.6), we deduce the following:

$$[v(a_2K)_y - a_2Kv_y - 2b_2Kv]_y$$

$$= v(a_2K)_{yy} - 2v(b_2K)_y - ka_2v_{yy} + 2b_2Kv_y. \tag{4.9}$$

Now observe that we have the following:

$$[v(a_3K)_z - a_3Kv_z - 2b_3Kv]_z$$

$$= v(a_3)_{zz}K + 2v(a_3)_zK_z + va_3K_{zz} - a_3Kv_{zz}$$

$$-2(b_3)_zKv - 2b_3K_zv - 2b_3Kv_z, \tag{4.10}$$

$$v(a_3K)_{zz} = v(a_3)_{zz}K + 2v(a_3)_zK_z + va_3K_{zz}, \tag{4.11}$$

$$v(b_3K)_z = vb_3K_z + v(b_3)_zK. \tag{4.12}$$

Now using (4.11) and (4.12) in (4.10), we deduce the following:

$$[v(a_3K)_z - a_3Kv_z - 2b_3Kv]_z$$

$$= v(a_3K)_{zz} - 2v(b_3K)_z - ka_3v_{zz} + 2b_3Kv_z. \tag{4.13}$$

Finally we have

$$[v(a_4K)_m - a_4Kv_m - 2b_4Kv]_m$$

$$= v(a_4)_{mm}K + 2v(a_4)_mK_m + va_4K_{mm} - a_4Kv_{mm}$$

$$-2(b_4)_mKv - 2b_4K_mv - 2b_4Kv_m, \tag{4.14}$$

$$v(a_4K)_{mm} = v(a_4)_{mm}K + 2v(a_4)_mK_m + va_4K_{mm}, \tag{4.15}$$

$$v(b_4K)_m = vb_4K_m + v(b_4)_mK. \tag{4.16}$$

Now using (4.15) and (4.16) in (4.14), we deduce the following:

$$[v(a_4K)_m - a_4Kv_m - 2b_4Kv]_m = v(a_4K)_{mm} - 2v(b_4K)_m$$

$$-ka_4v_{mm} + 2b_4Kv_m. \tag{4.17}$$

Thus the lemma follows by using (4.5), (4.9), (4.13), and (4.17) in (4.1).

4.2. The main theorem

The relationship between the first and second random walk problems is given by the following:

Theorem 4.2. *Suppose that the generalized rectangle* $[0, 1]\times[0, 1]\times[0, 1]\times[0, 1]$ *contains the point* $(\xi_1, \xi_2, \eta_1, \eta_2)$ *with*

$$v(0, y, z, m) = v(x, 0, z, m) = v(x, y, 0, m) = v(x, y, z, 0) = 0,$$

$$v(1, y, z, m) = v(x, 1, z, m) = v(x, y, 1, m) = v(x, y, z, 1) = 1,$$

$$K(0, y, z, m; \xi_1, \xi_2, \eta_1, \eta_2) = K(x, 0, z, m; \xi_1, \xi_2, \eta_1, \eta_2)$$

$$= K(x, y, 0, m; \xi_1, \xi_2, \eta_1, \eta_2) = K(x, y, z, 0; \xi_1, \xi_2, \eta_1, \eta_2) = 0,$$

$$K(1, y, z, m; \xi_1, \xi_2, \eta_1, \eta_2) = K(x, 1, z, m; \xi_1, \xi_2, \eta_1, \eta_2)$$

$$= K(x, y, 1, m; \xi_1, \xi_2, \eta_1, \eta_2) = K(x, y, z, 1; \xi_1, \xi_2, \eta_1, \eta_2) = 0.$$

Under this condition, the first and second random walk problems are related by

$$v(\xi_1, \xi_2, \eta_1, \eta_2) = 0.$$

Proof. We first integrate the expression in the conclusion of Lemma 4.1 over $[0, 1]\times[0, 1]\times[0, 1]\times[0, 1]$. Starting with the right hand-side of the expression in the conclusion of Lemma 4.1, we notice the following:

$$\int_0^1\int_0^1\int_0^1\int_0^1 [v(a_1K)_x - a_1Kv_x - 2b_1Kv]_x \, dxdydzdm$$

$$= \int_0^1\int_0^1\int_0^1 [v(a_1K)_x - a_1Kv_x - 2b_1Kv]_0^1 \, dydzdm$$

$$= \int_0^1\int_0^1\int_0^1 [(a_1K)_x]_0^1 \, dydzdm$$

$$= 0,$$

where the first equality follows from the Fundamental Theorem of

Calculus, the second equality follows from the assumption that $v(0, y, z, m) = 0$, $v(1, y, z, m) = 1$, $K(0, y, z, m; \xi_1, \xi_2, \eta_1, \eta_2) = K(1, y, z, m; \xi_1, \xi_2, \eta_1, \eta_2) = 0$, and the third equality follows by noting that $a_1 K$ is now a function of y, z, m thus $(a_1 K)_x = 0$. Now we also have the following:

$$\int_0^1\int_0^1\int_0^1\int_0^1 [v(a_2K)_y - a_2Kv_y - 2b_2Kv]_y \, dxdydzdm$$

$$= \int_0^1\int_0^1\int_0^1 [v(a_2K)_y - a_2Kv_y - 2b_2Kv]_0^1 \, dxdzdm$$

$$= \int_0^1\int_0^1\int_0^1 [(a_2K)_y]_0^1 \, dxdzdm$$

$$= 0.$$

Since the integrand is continuous, the first equality is an application of Fubini's Theorem, and for this, see Ref. [13] contained in [1] and the Fundamental Theorem of Calculus, the second equality follows from the assumption that $v(x, 0, z, m) = 0$, $v(x, 1, z, m) = 1$, $K(x, 0, z, m; \xi_1, \xi_2, \eta_1, \eta_2) = K(x, 1, z, m; \xi_1, \xi_2, \eta_1, \eta_2) = 0$, and the third equality follows by noting that a_2K is now a function of x, z, m thus $(a_2K)_y = 0$. Similarly, we deduce that

$$\int_0^1\int_0^1\int_0^1\int_0^1 [v(a_3K)_z - a_3Kv_z - 2b_3Kv]_z \, dxdydzdm = 0,$$

$$\int_0^1\int_0^1\int_0^1\int_0^1 [v(a_4K)_m - a_4Kv_m - 2b_4Kv]_m \, dxdydzdm = 0.$$

At this point it is clear that when we integrate the right hand side of the expression in Lemma 4.1 over $[0, 1]\times[0, 1]\times[0, 1]\times[0, 1]$, we get zero, thus it remains to show that

$$\int_0^1\int_0^1\int_0^1\int_0^1 (vL^*[K] - KL[v]) dxdydzdm = -v(\xi_1, \xi_2, \eta_1, \eta_2)$$

and the theorem follows. Now observe we have the following:

$$\begin{aligned}-v(\xi_1, \xi_2, \eta_1, \eta_2) &= -\int_0^1\int_0^1\int_0^1\int_0^1 \delta(x-\xi_1) \\ &\qquad \times \delta(y-\xi_2)\delta(z-\eta_1)\delta(m-\eta_2)dxdydzdm \\ &= \int_0^1\int_0^1\int_0^1\int_0^1 (vL^*[K] - KL[v])dxdydzdm.\end{aligned}$$

Note that the bottom equality in the above expression is a direct consequence of Lemma 4.1, Theorem 2.2, and Remark 3.4, and since $v(x, y, z, m)$ is a probability density function, which is continuous on $[0, 1]\times[0, 1]\times[0, 1]\times[0, 1]$ containing the point $(\xi_1, \xi_2, \eta_1, \eta_2)$, the conclusion of the Mean Value Property for Integrals, and for this, see Ref. [13] contained in [1] is easily verified for the first equality in the expression immediately above. Thus, $-v(\xi_1, \xi_2, \eta_1, \eta_2) = 0$, gives the desired result.

4.3. A discussion

With position-dependent jump probabilities one can deduce from the theorem in the previous section that there is no movement (zero flux) between particles in a neighborhood of the point $(\xi_1, \xi_2, \eta_1, \eta_2)$ and those near the boundary point $(\hat{x}, \hat{y}, \hat{z}, \hat{m})$. To see why notice that $v(\xi_1, \xi_2, \eta_1, \eta_2)$ characterizes the probability that the particle starting in a neighborhood of the point $(\xi_1, \xi_2, \eta_1, \eta_2)$ reaches the boundary point $(\hat{x}, \hat{y}, \hat{z}, \hat{m})$. Now consider the normal derivative $\partial K(\hat{x}, \hat{y}, \hat{z}, \hat{m}; \xi_1, \xi_2, \eta_1, \eta_2)/\partial n$ where K is the Green's function associated with the second random walk problem. This derivative can be physically interpreted as the flux of particle density associated with particles originating near the boundary point $(\hat{x}, \hat{y}, \hat{z}, \hat{m})$ and reaching the interior point $(\xi_1, \xi_2, \eta_1, \eta_2)$. Therefore, $-\partial K/\partial n$ is a measure of the flux in the reverse direction which should essentially equal

$\upsilon(\xi_1, \xi_2, \eta_1, \eta_2)$. However, $\upsilon(\xi_1, \xi_2, \eta_1, \eta_2) = 0$ from the theorem of the previous section, so the flux of the particle is essentially zero.

5. Result for the Third Random Walk Problem

5.1. The difference equation

Notice that the third random walk problem we consider determines the mean or expected time it takes for a particle starting at an interior point (x, y, z, m) in the prescribed region until it is absorbed at the boundary. Now assuming that the particle takes steps of length δ at intervals of time τ, and is equally likely to move to each of its eight neighbors, we deduce the following, upon letting $\mu(x, y, z, m)$ denote the mean

$$\begin{aligned}\mu(x, y, z, m) = \tau + [1 - \sum_{i=1}^{4}(p_i + q_i)]\mu(x, y, z, m)\\ +p_1\mu(x+\delta, y, z, m) + q_1\mu(x-\delta, y, z, m)\\ +p_2\mu(x, y+\delta, z, m) + q_2\mu(x, y-\delta, z, m)\\ +p_3\mu(x, y, z+\delta, m) + q_3\mu(x, y, z-\delta, m)\\ +p_4\mu(x, y, z, m+\delta) + q_4\mu(x, y, z, m-\delta). \qquad (5.1)\end{aligned}$$

In the above the time τ must be added since it signifies how long it takes the particle to reach one of its neighboring points in a single step.

5.2. The main theorem

Theorem 5.1. *Assume* $c_i(x, y, z, m) = \lim_{\delta,\tau\to 0} b_i\delta^2/4\tau$ *and* $D_i(x, y, z, m) = \lim_{\delta,\tau\to 0} a_i\delta^2/4\tau$ *for* $i = 1, 2, 3, 4$. *With the jump probabilities defined as in Equations* (1.1) *and* (1.2), *the limiting partial differential equation arising from* (5.1), *is given by the following, that is, the mean first passage time for each point,* $\mu(x, y, z, m)$, *satisfies the following*:

$$c_1\mu_x + c_2\mu_y + c_3\mu_z + c_4\mu_m + \frac{1}{2}[D_1\mu_{xx} + D_2\mu_{yy} + D_3\mu_{yy} + D_4\mu_{zz}] = -1.$$

The boundary condition for μ *is* $\mu(x, y, z, m) = 0$, $(x, y, z, m) \in \partial A$.

Proof. It may be completed along the lines of "Proof of Theorem III.2.13" [1], therefore we omit it.

Note that if $c_i(x, y, z, m) = 0$ and $D_i(x, y, z, m) = D$ for i=1, 2, 3, 4 in the above theorem, then we get the four-dimensional extension of Theorem II.5.1 [1].

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