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ON SOME EQUATIONS OF THE LAPLACIAN TYPE ARISING FROM DISCRETE RANDOM WALK PROBLEMS IN 7D

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Abstract

Let A be a bounded region in 7D, and let ∂A be its surface boundary which we assume to be absorbing. Enclose A and its boundary with a 7-parallelotope all of whose faces are possibly square, and let the sides be given by $x = a_i, a_{i+1}$ for $i = 1$; $y = a_i, a_{i+1}$ for $i = 3$; $z = a_i, a_{i+1}$ for $i = 5$; $m = a_i, a_{i+1}$ for $i = 7$; $r = a_i, a_{i+1}$ for $i = 9$; $v = a_i, a_{i+1}$ for $i = 11$; $j = a_i, a_{i+1}$ for $i = 13$. Let δ be the step length in the random walk, and assume that the intervals $[a_i, a_{i+1}]$ for $i = 1, 3, 5, 7, 9, 11, 13$ can be subdivided into the set of points

$$x_{k_1} = a_1 + \delta k_1,\ x_{n_1} = a_2,\ 0 \leq k_1 \leq n_1,$$

$$y_{k_2} = a_3 + \delta k_2,\ y_{n_2} = a_4,\ 0 \leq k_2 \leq n_2,$$

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$$z_{k_3} = a_5 + \delta k_3,\ z_{n_3} = a_6,\ 0 \le k_3 \le n_3,$$

$$m_{k_4} = a_7 + \delta k_4,\ m_{n_4} = a_8,\ 0 \le k_4 \le n_4,$$

$$r_{k_5} = a_9 + \delta k_5,\ r_{n_5} = a_{10},\ 0 \le k_5 \le n_5,$$

$$v_{k_6} = a_{11} + \delta k_6,\ v_{n_6} = a_{12},\ 0 \le k_6 \le n_6,$$

$$j_{k_7} = a_{13} + \delta k_7,\ j_{n_7} = a_{14},\ 0 \le k_7 \le n_7.$$

We say $(x_{k_1}, y_{k_2}, z_{k_3}, m_{k_4}, r_{k_5}, v_{k_6}, j_{k_7})$ is an interior point of A, if it does not lie on ∂A. If one of the neighboring points lies on ∂A or is exterior to A, we call it a boundary point. In this paper, working primarily in 7D, we study the following problems:

(a) What is the probability that a particle starting at an interior point (x, y, z, m, r, v, j) reaches the specified boundary point $(x_{w1}, y_{w2}, z_{w3}, m_{w4}, r_{w5}, v_{w6}, j_{w7})$ before it reaches and is absorbed at any other boundary point?

(b) What is the probability that a particle starting at an interior point (x, y, z, m, r, v, j) in the region A reaches a specified interior point $(\xi_1, \xi_2, \eta_1, \eta_2, \beta_1, \beta_2, \beta_3)$ before it reaches a boundary point and is absorbed?

(c) What is the "mean first passage time" for each point (x, y, z, m, r, v, j)?

1. Introduction

In this paper, the problems in the abstract are studied under the assumption that the time it takes the particle to reach the fixed boundary or interior point is not of concern, that is, the problems are investigated under the assumption of no time-dependence. We show in this paper, that such an investigation leads to some inhomogeneous forms of Laplace equation in 7D. We also give the boundary conditions along with its proof for solving Laplace equation in 7D. A significance of the discrete to continuous random walk derivations is the verification of some existence and uniqueness properties of these partial differential equations.

In Section 2, we consider the first random walk problem in the abstract in 7D, and show that the difference equation model for this problem when expanded in a Taylor series with remainder, and going in the limit leads to Laplace equation in 7D. This section takes inspiration from [1].

In Section 3, we prove the random walk process used in Section 2 for solving Laplace's equation in 7D. This section takes inspiration from [1], but more importantly [2].

In Section 4, we consider the second random walk problem in the abstract in 7D. In particular, we show the difference equation model for this problem when expanded in a Taylor series with remainder and going in the limit leads to an inhomogeneous form of Laplace's equation in 7D. If the particle starts its motion at an interior point which is not the one we fix, then this inhomogeneous form of Laplace's equation reduces to the homogeneous form of Laplace's equation in 7D. If the particle starts its motion at the fixed interior point, then this inhomogeneous form of Laplace's equation blows up. A consequence of the random walk formulation leads us to establish a relationship between the first two random walk problems in the abstract in 7D.

Finally, in Section 5, we consider the third random walk problem in the abstract in 7D, and show the difference equation model for this problem when expanded in a Taylor series with remainder and going in the limit leads to an equation of the Poisson type. This problem has been explored in the literature as the (mean) *first passage time* in the context of various phenomena of physical, biological, psychological and engineering importance. For examples, see Refs. [39], [12], [38], [37], [30], [31], [3], and [17] contained in [1]. Note that this problem is independent of time, since the possible times until absorption for each interior point is averaged out.

2. Derivation of Laplace Equation in 7D

We assume the particle is equally likely to move to any of its fourteen neighboring points from the point (x, y, z, m, r, v, j). Thus the probability it moves to any of its fourteen neighbors is $\frac{1}{14}$. The difference equation for the probability that the particle reaches the boundary point $(x_{w1}, y_{w2}, z_{w3}, m_{w4}, r_{w5}, v_{w6}, j_{w7})$ from the point (x, y, z, m, r, v, j) can be expressed in terms of the probability that it moves to any of its fourteen neighboring points and reaches $(x_{w1}, y_{w2}, z_{w3}, m_{w4}, r_{w5}, v_{w6}, j_{w7})$ from one of these points. Now let $f(x, y, z, m, r, v, j)$ be the probability the particle starts at the interior point (x, y, z, m, r, v, j) and reaches the boundary point $(x_{w1}, y_{w2}, z_{w3}, m_{w4}, r_{w5}, v_{w6}, j_{w7})$, then the difference equation is obtained as

$$\begin{aligned} f(x, y, z, m, r, v, j) = \frac{1}{14}[&f(x \pm \delta, y, z, m, r, v, j) + f(x, y \pm \delta, z, m, r, v, j) \\ &+ f(x, y, z \pm \delta, m, r, v, j) + f(x, y, z, m \pm \delta, r, v, j) \\ &+ f(x, y, z, m, r \pm \delta, v, j) + f(x, y, z, m, r, v \pm \delta, j) \\ &+ f(x, y, z, m, r, v, j \pm \delta)]. \end{aligned} \tag{2.1}$$

If (x, y, z, m, r, v, j) is a boundary point, we have

$$f(x, y, z, m, r, v, j)$$
$$= \begin{cases} 1, & \text{if } (x, y, z, m, r, v) = (x_{w1}, y_{w2}, z_{w3}, m_{w4}, r_{w5}, v_{w6}, j_{w7}), \\ 0, & \text{if } (x, y, z, m, r, v) \neq (x_{w1}, y_{w2}, z_{w3}, m_{w4}, r_{w5}, v_{w6}, j_{w7}). \end{cases} \tag{2.2}$$

Remark 2.1. Note that when the particle is at a boundary

$$(x, y, z, m, r, v) \neq (x_{w1}, y_{w2}, z_{w3}, m_{w4}, r_{w5}, v_{w6}, j_{w7})$$

it is absorbed and cannot reach the boundary point $(x_{w1}, y_{w2}, z_{w3}, m_{w4},$

$r_{w5}, v_{w6}, j_{w7})$. If one of the neighboring points of (x, y, z, m, r, v) is a boundary point, then (2.2) is to be used in (2.1).

Remark 2.2. In the limit as the step length $\delta \to 0$, the number of points in the subdivisions $[a_1, a_2], [a_3, a_4], [a_5, a_6], [a_7, a_8], [a_9, a_{10}], [a_{11}, a_{12}], [a_{13}, a_{14}]$, tend to infinity, and the boundary points defined actually lie on ∂A.

Now by Taylor's formula, one has the following expansions:

$$f(x \pm \delta, y, z, m, r, v, j) = f(x, y, z, m, r, v, j) \pm \delta f_x(x, y, z, m, r, v, j) + \frac{1}{2}\delta^2 f_{xx}(x, y, z, m, r, v, j) + O(\delta^3), \quad (2.3)$$

$$f(x, y \pm \delta, z, m, r, v, j) = f(x, y, z, m, r, v, j) \pm \delta f_y(x, y, z, m, r, v, j) + \frac{1}{2}\delta^2 f_{yy}(x, y, z, m, r, v, j) + O(\delta^3), \quad (2.4)$$

$$f(x, y, z \pm \delta, m, r, v, j) = f(x, y, z, m, r, v, j) \pm \delta f_z(x, y, z, m, r, v, j) + \frac{1}{2}\delta^2 f_{zz}(x, y, z, m, r, v, j) + O(\delta^3), \quad (2.5)$$

$$f(x, y, z, m \pm \delta, r, v, j) = f(x, y, z, m, r, v, j) \pm \delta f_m(x, y, z, m, r, v, j) + \frac{1}{2}\delta^2 f_{mm}(x, y, z, m, r, v, j) + O(\delta^3), \quad (2.6)$$

$$f(x, y, z, m, r \pm \delta, v, j) = f(x, y, z, m, r, v, j) \pm \delta f_r(x, y, z, m, r, v, j) + \frac{1}{2}\delta^2 f_{rr}(x, y, z, m, r, v, j) + O(\delta^3), \quad (2.7)$$

$$f(x, y, z, m, r, v \pm \delta, j) = f(x, y, z, m, r, v, j) \pm \delta f_v(x, y, z, m, r, v, j) + \frac{1}{2}\delta^2 f_{vv}(x, y, z, m, r, v, j) + O(\delta^3), \quad (2.8)$$

$$f(x, y, z, m, r, v, j \pm \delta) = f(x, y, z, m, r, v, j) \pm \delta f_j(x, y, z, m, r, v, j)$$

$$+\frac{1}{2}\delta^2 f_{jj}(x, y, z, m, r, v, j) + O(\delta^3). \qquad (2.9)$$

Now using (2.3)-(2.9) in (2.1) and simplifying gives

$$0 = \frac{1}{14}\delta^2[f_{xx}(x, y, z, m, r, v, j) + f_{yy}(x, y, z, m, r, v, j)$$

$$+f_{zz}(x, y, z, m, r, v, j) + f_{mm}(x, y, z, m, r, v, j)$$

$$+f_{rr}(x, y, z, m, r, v, j) + f_{vv}(x, y, z, m, r, v, j)$$

$$+f_{jj}(x, y, z, m, r, v, j)] + O(\delta^3). \qquad (2.10)$$

Multiplying (2.10) by $\frac{14}{\delta^2}$ and letting $\delta \to 0$ gives Laplace's equation in 7D

$$0 = f_{xx}(x, y, z, m, r, v, j) + f_{yy}(x, y, z, m, r, v, j)$$

$$+f_{zz}(x, y, z, m, r, v, j) + f_{mm}(x, y, z, m, r, v, j)$$

$$+f_{rr}(x, y, z, m, r, v, j) + f_{vv}(x, y, z, m, r, v, j)$$

$$+f_{jj}(x, y, z, m, r, v, j). \qquad (2.11)$$

The function $f(x, y, z, m, r, v, j)$ is now interpreted as a probability density. We further assume that the area Q is defined on ∂A and as $\delta \to 0$, the point $(x_{w1}, y_{w2}, z_{w3}, m_{w4}, r_{w5}, v_{w6}, j_{w7})$ tends to the (boundary) point $(\hat{x}, \hat{y}, \hat{z}, \hat{m}, \hat{r}, \hat{v}, \hat{j})$ on ∂A. The boundary conditions (2.2) now take the form

$$f(x, y, z, m, r, v, j)$$

$$= \{0, \text{ if } (x, y, z, m, r, v, j) \in \partial A,$$

$$(x, y, z, m, r, v, j) \notin (\hat{x}, \hat{y}, \hat{z}, \hat{m}, \hat{r}, \hat{v}, \hat{j})\}. \qquad (2.12)$$

Since $f(x, y, z, m, r, v, j)$ is now interpreted as a probability density, normalizing gives

$$\iint_{\partial A} f(x,\ y,\ z,\ m,\ r,\ v,\ j)dQ = 1. \tag{2.13}$$

Now we summarize the main result of this section as follows:

Theorem 2.3. *Assuming* $\delta \to 0$, *then the limiting partial differential equation of* (2.1) *is*

$$\begin{aligned} 0 = {} & f_{xx}(x,\ y,\ z,\ m,\ r,\ v,\ j) + f_{yy}(x,\ y,\ z,\ m,\ r,\ v,\ j) \\ & + f_{zz}(x,\ y,\ z,\ m,\ r,\ v,\ j) + f_{mm}(x,\ y,\ z,\ m,\ r,\ v,\ j) \\ & + f_{rr}(x,\ y,\ z,\ m,\ r,\ v,\ j) + f_{vv}(x,\ y,\ z,\ m,\ r,\ v,\ j) \\ & + f_{jj}(x,\ y,\ z,\ m,\ r,\ v,\ j) \end{aligned}$$

and the boundary condition for f is

$$\begin{aligned} & f(x,\ y,\ z,\ m,\ r,\ v,\ j) \\ = {} & \{0,\ if\ (x,\ y,\ z,\ m,\ r,\ v,\ j) \in \partial A, \\ & (x,\ y,\ z,\ m,\ r,\ v,\ j) \notin (\hat{x},\ \hat{y},\ \hat{z},\ \hat{m},\ \hat{r},\ \hat{v},\ \hat{j})\}. \end{aligned}$$

3. Proof of the Random Walk Method for Solving Laplace Equation in 7D

Our interest is to find the solution of Laplace's equation in a region in 7D bounded by parallel and perpendicular lines in each of the axial directions, given the value of the potential function at all points on the boundary. Suppose the value of the potential is required at some point A in the volume bounded by parallel and perpendicular lines in each of the axial directions. Put a marker at the point A. The random walk method operates as follows starting from A one can move into one of the fourteen directions in 7D. The direction chosen is random and such that each direction is equally likely. The probability of moving in any one direction is $\frac{1}{14}$. The choice of direction must be dependent on some suitable random process. Following [2], one could simulate the whole process by computer which could generate a random integer in the range 0 to 13 inclusive. A table such as the one below would then be used to interpret this number in terms of a move in the possible fourteen directions in 7D.

Table 1. Random integer generator

Number pair	Direction Pair
(0, 1)	(positive x-direction, negative x-direction)
(2, 3)	(positive y-direction, negative y-direction)
(4, 5)	(positive z-direction, negative z-direction)
(6, 7)	(positive m-direction, negative m-direction)
(8, 9)	(positive r-direction, negative r-direction)
(10, 11)	(positive v-direction, negative v-direction)
(12, 13)	(positive j-direction, negative j-direction)

Having made such one move the process is repeated until the marker at A reaches a point on ∂A. At this point the value of the potential say ξ_1 is noted. By this means a sequence $\{\xi_i\}$ is generated.

Theorem 3.1. *Let* ψ_A *denote the value of the potential at* A, *then*

$$\psi_A = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \xi_i}{n}$$

which states that the average value of the endpoint potentials $\{\xi_i\}$ *tends to* ψ_A, *as the number of random walks tends to infinity.*

Proof. Let B be an interior point. Denote the point $(p_1, p_2, q_1, q_2, r_1, r_2, r_3)$ as the intersection of the gridlines in 7D emanating from the axial directions. Now let $(p_1, p_2, q_1, q_2, r_1, r_2, r_3)$ be any point, $(i_1, i_2, j_1, j_2, k_1, k_2, k_3)$ be any interior point, and let $(a_1, a_2, s_1, s_2, f_1, f_2, f_3)$

be any boundary point. Let B be the point $(i_1, i_2, j_1, j_2, k_1, k_2, k_3)$, and let its potential be $\psi_{i_1, i_2, j_1, j_2, k_1, k_2, k_3}$, and consider the potentials $\psi_{i_1 \pm 1, i_2, j_1, j_2, k_1, k_2, k_3}$, that is, the potentials to the immediate right and left of B in the x-direction, then by Taylor series expansion of ψ in the positive and negative x-direction, we obtain

$$\psi_{i_1 \pm 1, i_2, j_1, j_2, k_1, k_2, k_3} = \psi_{i_1, i_2, j_1, j_2, k_1, k_2, k_3} \pm h\left[\frac{\partial \psi}{\partial x}\right]_B + \frac{h^2}{2}\left[\frac{\partial^2 \psi}{\partial x^2}\right]_B + O(h^3), \tag{3.1}$$

where h is the grid interval (step size). Now in the positive and negative part of each of the remaining axial directions, we obtain Taylor series expansion of ψ as follows

$$\psi_{i_1, i_2 \pm 1, j_1, j_2, k_1, k_2, k_3} = \psi_{i_1, i_2, j_1, j_2, k_1, k_2, k_3} \pm h\left[\frac{\partial \psi}{\partial y}\right]_B + \frac{h^2}{2}\left[\frac{\partial^2 \psi}{\partial y^2}\right]_B + O(h^3), \tag{3.2}$$

$$\psi_{i_1, i_2, j_1 \pm 1, j_2, k_1, k_2, k_3} = \psi_{i_1, i_2, j_1, j_2, k_1, k_2, k_3} \pm h\left[\frac{\partial \psi}{\partial z}\right]_B + \frac{h^2}{2}\left[\frac{\partial^2 \psi}{\partial z^2}\right]_B + O(h^3), \tag{3.3}$$

$$\psi_{i_1, i_2, j_1, j_2 \pm 1, k_1, k_2, k_3} = \psi_{i_1, i_2, j_1, j_2, k_1, k_2, k_3} \pm h\left[\frac{\partial \psi}{\partial m}\right]_B + \frac{h^2}{2}\left[\frac{\partial^2 \psi}{\partial m^2}\right]_B + O(h^3), \tag{3.4}$$

$$\psi_{i_1, i_2, j_1, j_2, k_1 \pm 1, k_2, k_3} = \psi_{i_1, i_2, j_1, j_2, k_1, k_2, k_3} \pm h\left[\frac{\partial \psi}{\partial r}\right]_B$$

$$+\frac{h^2}{2}\left[\frac{\partial^2\psi}{\partial r^2}\right]_B + O(h^3), \tag{3.5}$$

$$\psi_{i_1,i_2,j_1,j_2,k_1,k_2\pm 1,k_3} = \psi_{i_1,i_2,j_1,j_2,k_1,k_2,k_3} \pm h\left[\frac{\partial\psi}{\partial v}\right]_B$$

$$+\frac{h^2}{2}\left[\frac{\partial^2\psi}{\partial v^2}\right]_B + O(h^3), \tag{3.6}$$

$$\psi_{i_1,i_2,j_1,j_2,k_1,k_2,k_3\pm 1} = \psi_{i_1,i_2,j_1,j_2,k_1,k_2,k_3} \pm h\left[\frac{\partial\psi}{\partial j}\right]_B$$

$$+\frac{h^2}{2}\left[\frac{\partial^2\psi}{\partial j^2}\right]_B + O(h^3). \tag{3.7}$$

Now adding the right hand side of (3.1)-(3.7) gives

$$14\psi_{i_1,i_2,j_1,j_2,k_1,k_2,k_3}$$

$$+h^2\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial m^2}+\frac{\partial^2\psi}{\partial r^2}+\frac{\partial^2\psi}{\partial v^2}+\frac{\partial^2\psi}{\partial j^2}\right]_B + O(h^3). \tag{3.8}$$

On the other hand adding the left hand side of (3.1)-(3.3) gives

$$\psi_{i_1\pm 1,i_2,j_1,j_2,k_1,k_2,k_3} + \psi_{i_1,i_2\pm 1,j_1,j_2,k_1,k_2,k_3} + \psi_{i_1,i_2,j_1\pm 1,j_2,k_1,k_2,k_3} \tag{3.9}$$

and adding the left hand side of (3.4-3.7) gives

$$\psi_{i_1,i_2,j_1,j_2\pm 1,k_1,k_2,k_3} + \psi_{i_1,i_2,j_1,j_2,k_1\pm 1,k_2,k_3} + \psi_{i_1,i_2,j_1,j_2,k_1,k_2\pm 1,k_3}$$

$$+\psi_{i_1,i_2,j_1,j_2,k_1,k_2,k_3\pm 1}. \tag{3.10}$$

Since Laplace equation in 7D implies

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial m^2}+\frac{\partial^2\psi}{\partial r^2}+\frac{\partial^2\psi}{\partial v^2}+\frac{\partial^2\psi}{\partial j^2} = 0, \tag{3.11}$$

it follows that upon neglecting higher-order terms in (3.8) and equating the remaining term to the sum of (3.9) and (3.10), we get the following:

$$\begin{aligned}14\psi_{i_1,i_2,j_1,j_2,k_1,k_2,k_3} &= \psi_{i_1\pm1,i_2,j_1,j_2,k_1,k_2,k_3} + \psi_{i_1,i_2\pm1,j_1,j_2,k_1,k_2,k_3}\\ &+\psi_{i_1,i_2,j_1\pm1,j_2,k_1,k_2,k_3} + \psi_{i_1,i_2,j_1,j_2\pm1,k_1,k_2,k_3}\\ &+\psi_{i_1,i_2,j_1,j_2,k_1\pm1,k_2,k_3} + \psi_{i_1,i_2,j_1,j_2,k_1,k_2\pm1,k_3}\\ &+\psi_{i_1,i_2,j_1,j_2,k_1,k_2,k_3\pm1} \qquad (3.12)\end{aligned}$$

which is true for all interior points and therefore defines a set of equations with surface boundary conditions

$$\begin{aligned}\psi_{0,0,0,0,0,0,0} &= V_{0,0,0,0,0,0,0}\\ \psi_{1,0,0,0,0,0,0} &= V_{1,0,0,0,0,0,0}\\ &\vdots\\ &\text{etc.,} \qquad (3.13)\end{aligned}$$

where $V_{a1,a2,s1,s2,f1,f2,f3}$ represents the (known) values of ψ at the boundary point $(a1, a2, s1, s2, f1, f2, f3)$. The equations defining $\psi_{p_1,p_2,q_1,q_2,r_1,r_2,r_3}$ are thus

$$\begin{aligned}\psi_{i_1,i_2,j_1,j_2,k_1,k_2,k_3} &= \frac{1}{14}[\psi_{i_1\pm1,i_2,j_1,j_2,k_1,k_2,k_3} + \psi_{i_1,i_2\pm1,j_1,j_2,k_1,k_2,k_3}\\ &+\psi_{i_1,i_2,j_1\pm1,j_2,k_1,k_2,k_3} + \psi_{i_1,i_2,j_1,j_2\pm1,k_1,k_2,k_3}\\ &+\psi_{i_1,i_2,j_1,j_2,k_1\pm1,k_2,k_3} + \psi_{i_1,i_2,j_1,j_2,k_1,k_2\pm1,k_3}\\ &+\psi_{i_1,i_2,j_1,j_2,k_1,k_2,k_3\pm1}],\\ \psi_{a1,a2,s1,s2,f1,f2,f3} &= V_{a1,a2,s1,s2,f1,f2,f3}. \qquad (3.14)\end{aligned}$$

This completes the first part of the proof. Now let $P^{a1,a2,s1,s2,f1,f2,f3}_{i_1,i_2,j_1,j_2,k_1,k_2,k_3}$ be the probability of absorption at the boundary $(a1, a2, s1, s2, f1, f2, f3)$ starting from the interior point $(i_1, i_2, j_1, j_2, k_1, k_2, k_3)$, and executing a random walk of the type already described in this paper. Suppose that a reward $V_{a1,a2,s1,s2,f1,f2,f3}$ is associated with the absorption at $(a1, a2, s1, s2, f1, f2, f3)$ and let $E_{p_1,p_2,q_1,q_2,r_1,r_2,r_3}$ be the expected reward starting from the point $(p_1, p_2, q_1, q_2, r_1, r_2, r_3)$, that is, the average reward over a large number of trials. Consider the absorption at $(a1, a2, s1, s2, f1, f2, f3)$ starting from $(i_1, i_2, j_1, j_2, k_1, k_2, k_3)$. This can occur in one of 14 ways,

(a) a first move to $(i_1 \pm 1, i_2, j_1, j_2, k_1, k_2, k_3)$ and then absorption at $(a1, a2, s1, s2, f1, f2, f3)$ from there,

(b) a first move to $(i_1, i_2 \pm 1, j_1, j_2, k_1, k_2, k_3)$ and then absorption at $(a1, a2, s1, s2, f1, f2, f3)$ from there,

(c) a first move to $(i_1, i_2, j_1 \pm 1, j_2, k_1, k_2, k_3)$ and then absorption at $(a1, a2, s1, s2, f1, f2, f3)$ from there,

(d) a first move to $(i_1, i_2, j_1, j_2 \pm 1, k_1, k_2, k_3)$ and then absorption at $(a1, a2, s1, s2, f1, f2, f3)$ from there,

(e) a first move to $(i_1, i_2, j_1, j_2, k_1 \pm 1, k_2, k_3)$ and then absorption at $(a1, a2, s1, s2, f1, f2, f3)$ from there,

(f) a first move to $(i_1, i_2, j_1, j_2, k_1, k_2 \pm 1, k_3)$ and then absorption at $(a1, a2, s1, s2, f1, f2, f3)$ from there,

(g) a first move to $(i_1, i_2, j_1, j_2, k_1, k_2, k_3 \pm 1)$ and then absorption at $(a1, a2, s1, s2, f1, f2, f3)$ from there.

By elementary laws of probability, we have,

$$P^{a1,a2,s1,s2,f1,f2,f3}_{i_1,i_2,j_1,j_2,k_1,k_2,k_3} = \frac{1}{14}\Big[P^{a1,a2,s1,s2,f1,f2,f3}_{i_1\pm1,i_2,j_1,j_2,k_1,k_2,k_3} + P^{a1,a2,s1,s2,f1,f2,f3}_{i_1,i_2\pm1,j_1,j_2,k_1,k_2,k_3}$$

$$+ P^{a1,a2,s1,s2,f1,f2,f3}_{i_1,i_2,j_1\pm1,j_2,k_1,k_2,k_3} + P^{a1,a2,s1,s2,f1,f2,f3}_{i_1,i_2,j_1,j_2\pm1,k_1,k_2,k_3}$$

$$+ P^{a1, a2, s1, s2, f1, f2, f3}_{i_1, i_2, j_1, j_2, k_1 \pm 1, k_2, k_3} + P^{a1, a2, s1, s2, f1, f2, f3}_{i_1, i_2, j_1, j_2, k_1, k_2 \pm 1, k_3}$$

$$+ P^{a1, a2, s1, s2, f1, f2, f3}_{i_1, i_2, j_1, j_2, k_1, k_2, k_3 \pm 1}\Big]. \tag{3.15}$$

By definition of mathematical expectation

$$\sum_{a1, a2, s1, s2, f1, f2, f3} V_{a1, a2, s1, s2, f1, f2, f3} P^{a1, a2, s1, s2, f1, f2, f3}_{i_1, i_2, j_1, j_2, k_1, k_2, k_3} = E_{i_1, i_2, j_1, j_2, k_1, k_2, k_3}.$$

Now if we multiply (3.15) by $V_{a1, a2, s1, s2, f1, f2, f3}$ and sum over all possible points $(a1,\ a2,\ s1,\ s2,\ f1,\ f2,\ f3)$, then (3.15) which is valid for all interior points can be written as

$$E_{i_1, i_2, j_1, j_2, k_1, k_2, k_3} = \frac{1}{14}\Big[E_{i_1 \pm 1, i_2, j_1, j_2, k_1, k_2, k_3} + E_{i_1, i_2 \pm 1, j_1, j_2, k_1, k_2, k_3}$$

$$+ E_{i_1, i_2, j_1 \pm 1, j_2, k_1, k_2, k_3} + E_{i_1, i_2, j_1, j_2 \pm 1, k_1, k_2, k_3}$$

$$+ E_{i_1, i_2, j_1, j_2, k_1 \pm 1, k_2, k_3} + E_{i_1, i_2, j_1, j_2, k_1, k_2 \pm 1, k_3}$$

$$+ E_{i_1, i_2, j_1, j_2, k_1, k_2, k_3 \pm 1}\Big]. \tag{3.16}$$

At the boundary points we have $P^{a1, a2, s1, s2, f1, f2, f3}_{a1, a2, s1, s2, f1, f2, f3} = 1$ since there is a unit probability of absorption at the boundary point $(a1,\ a2,\ s1,\ s2,\ f1,\ f2,\ f3)$ considering we are already there to begin with. Thus at boundary points we have

$$E_{a1, a2, s1, s2, f1, f2, f3} = V_{a1, a2, s1, s2, f1, f2, f3} P^{a1, a2, s1, s2, f1, f2, f3}_{a1, a2, s1, s2, f1, f2, f3}$$

$$= V_{a1, a2, s1, s2, f1, f2, f3}. \tag{3.17}$$

Thus, $E_{p_1, p_2, q_1, q_2, r_1, r_2, r_3}$ satisfies the equations

$$E_{i_1, i_2, j_1, j_2, k_1, k_2, k_3} = \frac{1}{14}\Big[E_{i_1 \pm 1, i_2, j_1, j_2, k_1, k_2, k_3} + E_{i_1, i_2 \pm 1, j_1, j_2, k_1, k_2, k_3}$$

$$+E_{i_1,i_2,j_1\pm1,j_2,k_1,k_2,k_3}+E_{i_1,i_2,j_1,j_2\pm1,k_1,k_2,k_3}$$

$$+E_{i_1,i_2,j_1,j_2,k_1\pm1,k_2,k_3}+E_{i_1,i_2,j_1,j_2,k_1,k_2\pm1,k_3}$$

$$+E_{i_1,i_2,j_1,j_2,k_1,k_2,k_3\pm1}],$$

$$E_{a1,a2,s1,s2,f1,f2,f3}=V_{a1,a2,s1,s2,f1,f2,f3}. \tag{3.18}$$

A comparison of (3.14) and (3.18) shows that they are in fact identical with $E_{p_1,p_2,q_1,q_2,r_1,r_2,r_3}$ in place of $\psi_{p_1,p_2,q_1,q_2,r_1,r_2,r_3}$. A solution of $E_{p_1,p_2,q_1,q_2,r_1,r_2,r_3}$ is therefore a solution of $\psi_{p_1,p_2,q_1,q_2,r_1,r_2,r_3}$, and it is $E_{p_1,p_2,q_1,q_2,r_1,r_2,r_3}$ that is found experimentally by the procedure described at the beginning of this section.

4. An Inhomogeneous forms of Laplace Equation in 7D

Let (x, y, z, m, r, v) be an interior point in the second random walk problem of the abstract, and let $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$ be a fixed interior point. Since the problem is time independent, and the particle does not stop its motion once it reaches $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$ for the first time, it is possible for the particle to pass through the point $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$ more than once before it reaches and is absorbed at the boundary. Consequently, if the particle begins its motion at $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$, it has unit probability of reaching $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$, since it is already there to begin with. However, it can also move to one of its 14 neighboring points and each $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$ from there if the neighbor is not a boundary point. Therefore, if we introduce a function $w(x, y, z, m, r, v)$ that characterizes the prospects of a particle reaching $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$, from the starting point (x, y, z, m, r, v), we cannot consider $w(x, y, z, m, r, v)$ to be a probability distribution since it may assume values exceeding unity. In particular, $w(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3) \geq 1.$

4.1. The difference equation for $w(x, y, z, m, r, v)$

Case (I). If (x, y, z, m, r, v) is a boundary point, then

$$w(x, y, z, m, r, v) = 0. \tag{4.1}$$

Case (II). If $(x, y, z, m, r, v) \neq (\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$, then

$$\begin{aligned} w(x, y, z, m, r, v) = \frac{1}{14}[&w(x \pm \delta, y, z, m, r, v) \\ &+ w(x, y \pm \delta, z, m, r, v) + w(x, y, z \pm \delta, m, r, v) \\ &+ w(x, y, z, m \pm \delta, r, v) + w(x, y, z, m, r \pm \delta, v) \\ &+ w(x, y, z, m, r, v \pm \delta)]. \end{aligned} \tag{4.2}$$

Case (III). If $(x, y, z, m, r, v) = (\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$, then

$$\begin{aligned} &w(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3) \\ &= 1 + \frac{1}{14}[w(\xi_1 \pm \delta, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3) \\ &\quad + w(\xi_1, \xi_2 \pm \delta, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3) + w(\xi_1, \xi_2, \eta_1 \pm \delta, \eta_2, \gamma_1, \gamma_2, \gamma_3) \\ &\quad + w(\xi_1, \xi_2, \eta_1, \eta_2 \pm \delta, \gamma_1, \gamma_2, \gamma_3) + w(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1 \pm \delta, \gamma_2, \gamma_3) \\ &\quad + w(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2 \pm \delta, \gamma_3) \\ &\quad + w(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3 \pm \delta)]. \end{aligned} \tag{4.3}$$

4.2. Taylor expansions in the right hand side of (4.2)

$$\begin{aligned} &w(x \pm \delta, y, z, m, r, v, j) \\ &= w(x, y, z, m, r, v, j) \pm \delta w_x(x, y, z, m, r, v, j) \\ &\quad + \frac{1}{2}\delta^2 w_{xx}(x, y, z, m, r, v, j) + O(\delta^3), \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
& w(x,\ y \pm \delta,\ z,\ m,\ r,\ v,\ j) \\
= {} & w(x,\ y,\ z,\ m,\ r,\ v,\ j) \pm \delta w_y(x,\ y,\ z,\ m,\ r,\ v,\ j) \\
& + \frac{1}{2}\delta^2 w_{yy}(x,\ y,\ z,\ m,\ r,\ v,\ j) + O(\delta^3),
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
& w(x,\ y,\ z \pm \delta,\ m,\ r,\ v,\ j) \\
= {} & w(x,\ y,\ z,\ m,\ r,\ v,\ j) \pm \delta w_z(x,\ y,\ z,\ m,\ r,\ v,\ j) \\
& + \frac{1}{2}\delta^2 w_{zz}(x,\ y,\ z,\ m,\ r,\ v,\ j) + O(\delta^3),
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
& w(x,\ y,\ z,\ m \pm \delta,\ r,\ v,\ j) \\
= {} & w(x,\ y,\ z,\ m, r,\ v,\ j) \pm \delta w_m(x,\ y,\ z,\ m, r,\ v,\ j) \\
& + \frac{1}{2}\delta^2 w_{mm}(x,\ y,\ z,\ m,\ r,\ v,\ j) + O(\delta^3),
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
& w(x,\ y,\ z,\ m,\ r \pm \delta,\ v,\ j) \\
= {} & w(x,\ y,\ z,\ m, r,\ v,\ j) \pm \delta w_r(x,\ y,\ z,\ m, r,\ v,\ j) \\
& + \frac{1}{2}\delta^2 w_{rr}(x,\ y,\ z,\ m,\ r,\ v,\ j) + O(\delta^3),
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
& w(x,\ y,\ z,\ m,\ r,\ v \pm \delta,\ j) \\
= {} & w(x,\ y,\ z,\ m,\ r,\ v,\ j) \pm \delta w_v(x,\ y,\ z,\ m, r,\ v,\ j) \\
& + \frac{1}{2}\delta^2 w_{vv}(x,\ y,\ z,\ m,\ r,\ v,\ j) + O(\delta^3),
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
& w(x,\ y,\ z,\ m,\ r,\ v,\ j \pm \delta) \\
= {} & w(x,\ y,\ z,\ m,\ r,\ v,\ j) \pm \delta w_j(x,\ y,\ z,\ m, r,\ v,\ j) \\
& + \frac{1}{2}\delta^2 w_{jj}(x,\ y,\ z,\ m,\ r,\ v,\ j) + O(\delta^3).
\end{aligned} \tag{4.10}
$$

4.3. The main theorem

Theorem 4.1. *For small* δ, *the inhomogeneous form of Laplace equation arising from* (4.2) *and* (4.3) *is given by*

$$w_{xx} + w_{yy} + w_{zz} + w_{mm} + w_{rr} + w_{vv} + w_{jj}$$

$$= \begin{cases} O(\delta), & \text{if } (x, y, z, m, r, v) \neq (\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3), \\ \dfrac{-14}{\delta^2} + O(\delta), & \text{if } (x, y, z, m, r, v) = (\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3). \end{cases}$$

Proof. We consider two cases.

Case (I). We assume $(x, y, z, m, r, v) \neq (\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$.

Now substituting (4.4)-(4.10) into (4.2) and simplifying gives

$$O(\delta^3) = \frac{\delta^2}{14} w_{xx} + \frac{\delta^2}{14} w_{yy} + \frac{\delta^2}{14} w_{zz} + \frac{\delta^2}{14} w_{mm} + \frac{\delta^2}{14} w_{rr} + \frac{\delta^2}{14} w_{vv}. \tag{4.11}$$

Now multiplying (4.11) by $\frac{14}{\delta^2}$ gives the first part of the theorem.

Case (II). We assume $(x, y, z, m, r, v) = (\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$.

Again substituting (4.4)-(4.10) into (4.3) and simplifying gives

$$O(\delta^3) = 1 + \frac{\delta^2}{14} w_{xx} + \frac{\delta^2}{14} w_{yy} + \frac{\delta^2}{14} w_{zz} + \frac{\delta^2}{14} w_{mm} + \frac{\delta^2}{14} w_{rr} + \frac{\delta^2}{14} w_{vv}. \tag{4.12}$$

Now multiplying (4.12) by $\frac{14}{\delta^2}$ and subtracting one from both sides of the equality gives the second part of the theorem, and the proof is complete.

Remark 4.2. From Theorem 4.1, it is clear that if $\delta \to 0$ and

$(x, y, z, m, r, v) = (\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$, we have blow up. On the other hand if $\delta \to 0$ and $(x, y, z, m, r, v) \neq (\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$, then we recover the homogeneous form of Laplace's equation in 7D.

5. Connecting the First and Second Random Walk Problems

5.1. A seven dimensional Dirac delta function with singular point

If $\delta \to 0$ in Theorem 4.1, then $w(x, y, z, m, r, v, j)$ is a density function, and thus the property of the integral of $w(x, y, z, m, r, v, j)$ over a small neighborhood of (x, y, z, m, r, v, j) is of most interest. Now consider the 7-parallelotope with center at $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$ and with sides proportional to the square of the step length δ, that is, the sides of the 7-parallelotope have area proportional to δ^2. The total surface area of the 7-parallelotope is then proportional to $14\delta^2$, this quantity multiplied by $\frac{14}{\delta^2}$, tends to a finite nonzero limit as $\delta \to 0$. Since the right hand side of the expression in Theorem 4.1 vanishes for $(x, y, z, m, r, v, j) \neq (\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$, and its integral over the 7-parallelotope has a finite nonzero limit as $\delta \to 0$, then in the limit $w(x, y, z, m, r, v, j)$ must behave like a 7-dimensional extension of the Dirac Delta function with singular point $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$ which we introduce as follows:

$$\delta(x - \xi_1)\delta(y - \xi_2)\delta(z - \eta_1)\delta(m - \eta_2)\delta(r - \gamma_1)\delta(v - \gamma_2)\delta(j - \gamma_3) = 0,$$

$$(x, y, z, m, r, v, j) \neq (\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3),$$

$$\int\int\cdots\int\int_R \delta(x - \xi_1)\delta(y - \xi_2)\delta(z - \eta_1)\delta(m - \eta_2)\delta(r - \gamma_1)\delta(v - \gamma_2)\delta(j - \gamma_3)$$

$$\times\, dxdydzdmdrdvdj = 1,$$

where R is any open region in the 7-parallelotope containing the point $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$.

5.2. A mean value type property

Theorem 5.1. *Let* $f(x, y, z, m, r, v, j)$ *be continuous on the 7-parallelotope with center at* $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$, *then we have*

$$f(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$$

$$= \int\int\cdots\int\int_R f(x, y, z, m, r, v, j)\delta(x - \xi_1)\delta(y - \xi_2)\delta(z - \eta_1)$$

$$\times \delta(m - \eta_2)\delta(r - \gamma_1)\delta(v - \gamma_2)\delta(j - \gamma_3)dxdydzdmdrdvdj.$$

Proof. Let $\epsilon > 0$ be given. Recall from [1] the Dirac delta function, $\delta(x)$ can be defined as the limit of a sequence of discontinuous functions, $\delta_\epsilon(x)$, where

$$\delta_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon}, & \text{if } |x| < \epsilon, \\ 0, & \text{if } |x| > \epsilon. \end{cases}$$

From this definition, we see that by shifting $\delta_\epsilon(x)$ to the right by ξ_1 units, we can define

$$\delta_\epsilon(x - \xi_1) = \begin{cases} \frac{1}{2\epsilon}, & \text{if } |x - \xi_1| < \epsilon, \\ 0, & \text{if } |x - \xi_1| > \epsilon. \end{cases}$$

A similar definition holds for

$\delta_\epsilon(y - \xi_2)$, $\delta_\epsilon(z - \eta_1)$, $\delta_\epsilon(m - \eta_2)$, $\delta_\epsilon(r - \gamma_1)$, $\delta_\epsilon(v - \gamma_2)$, and $\delta_\epsilon(j - \gamma_3)$.

Now let

$$R = [-\epsilon + \xi_1, \epsilon + \xi_1]\times[-\epsilon + \xi_2, \epsilon + \xi_2]\times[-\epsilon + \eta_1, \epsilon + \eta_1]$$

$$\times [-\epsilon + \eta_2, \epsilon + \eta_2]\times[-\epsilon + \gamma_1, \epsilon + \gamma_1]\times[-\epsilon + \gamma_2, \epsilon + \gamma_2]$$

$$\times [-\epsilon + \gamma_3, \epsilon + \gamma_3]$$

which contains the point $(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$, then from Ref. [13] contained in [1], we see that R is a compact, path-wise connected Jordan domain in $\mathbb{R}^7$ with positive volume $128\epsilon^7$. Since f is continuous on R, then invoking the Mean Value Property for Integrals in Ref. [13] contained in [1], we deduce the following:

$$f(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$$

$$= \lim_{\epsilon \to 0} \iint \cdots \iint_R f(x, y, z, m, r, v, j) \frac{1}{128\epsilon^7} dxdydzdmdrdvdj$$

$$= \lim_{\epsilon \to 0} \iint \cdots \iint_R f(x, y, z, m, r, v, j) \delta_\epsilon(x - \xi_1)\delta_\epsilon(y - \xi_2)\delta_\epsilon(z - \eta_1)$$

$$\times \delta_\epsilon(m - \eta_2)\delta_\epsilon(r - \gamma_1)\delta_\epsilon(v - \gamma_2)\delta_\epsilon(j - \gamma_3) dxdydzdmdrdvdj$$

$$= \iint \cdots \iint_R f(x, y, z, m, r, v, j)\delta(x - \xi_1)\delta(y - \xi_2)\delta(z - \eta_1)$$

$$\times \delta(m - \eta_2)\delta(r - \gamma_1)\delta(v - \gamma_2)\delta(j - \gamma_3) dxdydzdmdrdvdj.$$

5.3. A green type function with homogeneous boundary conditions

Observe that as $\delta \to 0$, the boundary points of the discrete problem tend to points on ∂A, and if (x, y, z, m, r, v, j) is a boundary point then $w(x, y, z, m, r, v, j)$ vanishes on ∂A. To analyze the properties of the solution of this boundary problem, we replace $w(x, y, z, m, r, v, j)$ with the following, which is an extension of the Green's function, $K(x, y, z, m, r, v, j; \xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$. In particular, this so-called Green's function for Laplace's equation in 7D with homogeneous boundary conditions, we define to be a solution of

$$\nabla^2 K = -\delta(x - \xi_1)\delta(y - \xi_2)\delta(z - \eta_1)\delta(m - \eta_2)\delta(r - \gamma_1)$$

$$\times \delta(v - \gamma_2)\delta(j - \gamma_3), \quad (x, y, z, m, r, v, j) \in \partial A$$

which satisfies the boundary condition

$$K(x, y, z, m, r, v, j; \xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3) = 0, (x, y, z, m, r, v, j) \in \partial A.$$

Note that

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$+ \frac{\partial^2}{\partial m^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial j^2}.$$

5.4. The main theorem

Consider the Green's function $K(x, y, z, m, r, v, j; \xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$ associated with the second random walk problem and the function $f(x, y, z, m, r, v, j)$ associated with the first random walk problem. Now consider the seven dimensional extension of Green's second identity and apply it to both functions, integrating over the region A and its boundary ∂A, gives

$$\iint \cdots \iint_A \{f\nabla^2 K - K\nabla^2 f\} dxdydzdmdrdvdj$$

$$= \iint_A \left\{ f \frac{\partial K}{\partial n} - K \frac{\partial f}{\partial n} \right\} dQ, \tag{5.1}$$

where $\frac{\partial}{\partial n}$ is a derivative normal to ∂A, and Q is an area on ∂A, and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$+ \frac{\partial^2}{\partial m^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial j^2}.$$

Now the connection between the first two random walk problems is given as follows:

Theorem 5.2. *The density function* $f(x, y, z, m, r, v, j)$ *for the first random walk problem, and the Green's function* $K(x, y, z, m, r, v, j; \xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$ *for the second random walk problem are related by*

$$f(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)$$

$$= -\frac{\partial K(\hat{x}, \hat{y}, \hat{z}, \hat{m}, \hat{r}, \hat{v}, \hat{j}; \xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)}{\partial n}.$$

Proof. Observe from (2.11) that $\nabla^2 f = 0$, and from the Green's function for Laplace equation in 7D,

$$\nabla^2 K = -\delta(x - \xi_1)\delta(y - \xi_2)\delta(z - \eta_1)\delta(m - \eta_2)\delta(r - \gamma_1)\delta(v - \gamma_2)\delta(j - \gamma_3).$$

So upon using Theorem 5.1, the left hand side of (5.1) becomes

$$\iint\cdots\iint_A \{f\nabla^2 K - K\nabla^2 f\} dxdydzdmdrdvdj$$

$$= -\iint\cdots\iint_A f(x, y, z, m, r, v, j)$$

$$\times\, \delta(x - \xi_1)\delta(y - \xi_2)\delta(z - \eta_1)\delta(m - \eta_2)$$

$$\times\, \delta(r - \gamma_1)\delta(v - \gamma_2)\delta(j - \gamma_3) dxdydzdmdrdvdj$$

$$= -f(\xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3). \tag{5.2}$$

Now from (2.12) and (2.13), it follows on the boundary we have

$$f(x, y, z, m, r, v, j)$$

$$\times \frac{\partial K(x, y, z, m, r, v, j; \xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)}{\partial n}.$$

$$= f(x, y, z, m, r, v, j)$$

$$\times \frac{\partial K(\hat{x}, \hat{y}, \hat{z}, \hat{m}, \hat{r}, \hat{v}, \hat{j}; \xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)}{\partial n} \tag{5.3}$$

and since K vanishes on the boundary, the right hand side of (5.1) now becomes

$$\iint_A f(x, y, z, m, r, v, j)$$

$$\times \frac{\partial K(\hat{x}, \hat{y}, \hat{z}, \hat{m}, \hat{r}, \hat{v}, \hat{j}; \xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)}{\partial n} dQ$$

$$= \frac{\partial K(\hat{x}, \hat{y}, \hat{z}, \hat{m}, \hat{r}, \hat{v}, \hat{j}; \xi_1, \xi_2, \eta_1, \eta_2, \gamma_1, \gamma_2, \gamma_3)}{\partial n}.$$

Thus by equating the right hand side of (5.2) to the right hand side of the equation immediately above, the result follows.

6. An Equation of the Poisson Type

6.1. The difference equation for the mean first passage time

We assume the particle takes steps of length δ at intervals of time τ, and is equally likely to move to each of its 14 neighboring points from the point (x, y, z, m, r, v, j), thus the mean first passage time for each point, which we denote by $N(x, y, z, m, r, v, j)$, obeys the following difference equation

$$N(x, y, z, m, r, v, j)$$

$$= \tau + \frac{1}{14}[N(x \pm \delta, y, z, m, r, v, j)$$

$$+N(x, y \pm \delta, z, m, r, v, j) + N(x, y, z \pm \delta, m, r, v, j)$$

$$+N(x, y, z, m \pm \delta, r, v, j) + N(x, y, z, m, r \pm \delta, v, j)$$

$$+N(x, y, z, m, r, v \pm \delta, j) + N(x, y, z, m, r, v, j \pm \delta)]. \tag{6.1}$$

Note that the above equation expresses the expected time until absorption, $N(x, y, z, m, r, v, j)$, in terms of the expected time until absorption for each of its 14 points, multiplied by the probability $\frac{1}{14}$ that the particle moves to each of these points. The time τ is added as it signifies how long it takes for the particle to reach one of the neighboring points in a single step.

6.2. Taylor expansions in the right hand side of (6.1)

Now by Taylor's formula, one has the following expansions:

$$N(x \pm \delta, y, z, m, r, v, j)$$
$$= N(x, y, z, m, r, v, j) \pm \delta N_x(x, y, z, m, r, v, j)$$
$$+ \frac{1}{2}\delta^2 N_{xx}(x, y, z, m, r, v, j) + O(\delta^3), \tag{6.2}$$

$$N(x, y \pm \delta, z, m, r, v, j)$$
$$= N(x, y, z, m, r, v, j) \pm \delta N_y(x, y, z, m, r, v, j)$$
$$+ \frac{1}{2}\delta^2 N_{yy}(x, y, z, m, r, v, j) + O(\delta^3), \tag{6.3}$$

$$N(x, y, z \pm \delta, m, r, v, j)$$
$$= N(x, y, z, m, r, v, j) \pm \delta N_z(x, y, z, m, r, v, j)$$
$$+ \frac{1}{2}\delta^2 N_{zz}(x, y, z, m, r, v, j) + O(\delta^3), \tag{6.4}$$

$$N(x, y, z, m \pm \delta, r, v, j)$$
$$= N(x, y, z, m, r, v, j) \pm \delta N_m(x, y, z, m, r, v, j)$$
$$+ \frac{1}{2}\delta^2 N_{mm}(x, y, z, m, r, v, j) + O(\delta^3), \tag{6.5}$$

$$N(x, y, z, m, r \pm \delta, v, j)$$
$$= N(x, y, z, m, r, v, j) \pm \delta N_r(x, y, z, m, r, v, j)$$
$$+ \frac{1}{2}\delta^2 N_{rr}(x, y, z, m, r, v, j) + O(\delta^3), \tag{6.6}$$

$$N(x, y, z, m, r, v \pm \delta, j)$$

$$= N(x, y, z, m, r, v, j) \pm \delta N_v(x, y, z, m, r, v, j)$$

$$+ \frac{1}{2}\delta^2 N_{vv}(x, y, z, m, r, v, j) + O(\delta^3), \tag{6.7}$$

$$N(x, y, z, m, r, v, j \pm \delta)$$

$$= N(x, y, z, m, r, v, j) \pm \delta N_j(x, y, z, m, r, v, j)$$

$$+ \frac{1}{2}\delta^2 N_{jj}(x, y, z, m, r, v, j) + O(\delta^3). \tag{6.8}$$

6.3. The main theorem

Theorem 6.1. *Assume* $D = \lim_{\delta\to 0, \tau\to 0} \frac{\delta^2}{7\tau}$, *then the limiting partial differential equation arising from* (6.1) *is given by*

$$\frac{1}{2}D[N_{xx} + N_{yy} + N_{zz} + N_{mm} + N_{rr} + N_{vv} + N_{jj}] = -1.$$

The boundary condition for N *is given by*

$$N(x, y, z, m, r, v, j) = 0, \quad (x, y, z, m, r, v, j) \in \partial A.$$

Proof. Substitute (6.2)-(6.8) into (6.1) and simplifying gives

$$0 = \tau + \frac{1}{14}\delta^2 N_{xx} + \frac{1}{14}\delta^2 N_{yy} + \frac{1}{14}\delta^2 N_{zz} + \frac{1}{14}\delta^2 N_{mm}$$

$$+ \frac{1}{14}\delta^2 N_{rr} + \frac{1}{14}\delta^2 N_{vv} + \frac{1}{14}\delta^2 N_{jj} + O(\delta^3). \tag{6.9}$$

Now multiplying (6.9) by $\frac{14}{7\tau}$ gives

$$0 = 2 + \frac{1}{7\tau}\delta^2 N_{xx} + \frac{1}{7\tau}\delta^2 N_{yy} + \frac{1}{7\tau}\delta^2 N_{zz} + \frac{1}{7\tau}\delta^2 N_{mm}$$

$$+ \frac{1}{7\tau}\delta^2 N_{rr} + \frac{1}{7\tau}\delta^2 N_{vv} + \frac{1}{7\tau}\delta^2 N_{jj} + O(\delta^3). \tag{6.10}$$

By assumption $D = \lim_{\delta\to 0, \tau\to 0} \frac{\delta^2}{7\tau}$ and applying this assumption to (6.10), multiplying the result by $\frac{1}{2}$, and keeping the term $\frac{1}{2}D[N_{xx} + N_{yy} + N_{zz} + N_{mm} + N_{rr} + N_{vv} + N_{jj}]$ on the left hand side of the equation gives the desired result. Now if (x, y, z, m, r, v, j) is a boundary point, then, $N(x, y, z, m, r, v, j) = 0$, since the time until absorption at any boundary point is zero. ,

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