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ON AN ASYMPTOTIC RELATIONSHIP BETWEEN BETA AND GAMMA DISTRIBUTIONS

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In this paper, we show that gamma distributions can be considered as weak limits of appropriately rescaled beta distributions.

Let X be a random variable with a standard beta distribution, i.e., with a p.d.f. f given by

$$f = (x, \alpha, \beta) = \frac{x^{\alpha-1} \cdot (1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1; \quad \alpha, \beta > 0,$$

where $B(\alpha, \beta) = \frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta)}$ denotes Euler's Beta function. The first moments are given by $E(X) = \frac{\alpha}{\alpha+\beta}$, $Var(X) = \frac{\alpha \cdot \beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ (cf. [2], relations (25.15a) and (25.15b), p. 217).

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Then for $A > 0$, the p.d.f. of $A \cdot X$ is clearly given by

$$\frac{1}{A} f\left(\frac{x}{A};\ \alpha,\ \beta\right) = \frac{x^{\alpha-1} \cdot \left(1 - \frac{x}{A}\right)^{\beta-1}}{A^{\alpha} \cdot B(\alpha,\ \beta)} = \frac{x^{\alpha-1} \cdot \left(1 - \frac{x}{A}\right)^{A \cdot \frac{\beta-1}{A}}}{A^{\alpha} \cdot B(\alpha,\ \beta)},$$

$$0 < x < 1; \quad \alpha,\ \beta > 0.$$

For the first moments, we thus obtain

$$E(A \cdot X) = A \cdot \frac{\alpha}{\alpha+\beta}, \quad Var(A \cdot X) = A^2 \cdot \frac{\alpha \cdot \beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Further, we consider a gamma distributed random variable Y with a p.d.f. given by $g(x;\ \alpha,\ \lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x)$, $x > 0$; $\alpha,\ \lambda > 0$ (cf. [1], relation (17.23), p. 343, with $\lambda = \frac{1}{\beta}$ there) with moments $E(Y) = \frac{\alpha}{\lambda}$, $Var(Y) = \frac{\alpha}{\lambda^2}$ (follows immediately from [1], relation (17.8), p. 339 and [1], relation (17.23), p. 343). Equating expectations for $A \cdot X$ and Y, we obtain $\beta = A \cdot \lambda - \alpha$ or $\lambda = \frac{\alpha+\beta}{A}$. Thus the p.d.f. of $A \cdot X$ becomes

$$\frac{1}{A} f\left(\frac{x}{A};\ \alpha,\ \beta\right) = \frac{x^{\alpha-1} \cdot \left(1 - \frac{x}{A}\right)^{\beta-1}}{A^{\alpha} \cdot B(\alpha,\ \beta)} = \frac{x^{\alpha-1} \cdot \left(1 - \frac{x}{A}\right)^{A \cdot \lambda}}{A^{\alpha} \cdot B(\alpha,\ \beta) \cdot \left(1 - \frac{x}{A}\right)^{1+\alpha}}$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot \left(1 - \frac{x}{A}\right)^{A \cdot \lambda} \cdot \frac{\Gamma(\alpha+\beta)}{(\alpha+\beta)^2 \cdot \Gamma(\beta) \cdot \left(1 - \frac{x}{A}\right)^{1+\alpha}}$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot \left(1 - \frac{x}{A}\right)^{A\cdot\lambda} \cdot \frac{\Gamma(A \cdot \lambda)}{(A \cdot \lambda)^{\alpha} \cdot \Gamma(A \cdot \lambda - \alpha) \cdot \left(1 - \frac{x}{A}\right)^{1+\alpha}},$$

$$0 < x < 1; \;\; \alpha, \lambda > 0.$$

Now, for $A \to \infty$, we get $\left(1 - \frac{x}{A}\right)^{A\cdot\lambda} \to \exp(-\lambda x)$, $\left(1 - \frac{x}{A}\right)^{1+\alpha} \to 1$ and

$$\frac{\Gamma(A \cdot \lambda)}{(A \cdot \lambda)^{\alpha} \cdot \Gamma(A \cdot \lambda - \alpha)} \to 1 \qquad (*)$$

which means that the beta distribution of $A \cdot X$ with parameters α and β tends weakly to the gamma distribution with parameters α and λ. Note that for integer α, we have, by the recursive representation of the gamma function (cf. [3], 11.1.2.1, p. 222),

$$\Gamma(A \cdot \lambda) = \Gamma(A \cdot \lambda - 1) \cdot (A \cdot \lambda - 1) = \Gamma(A \cdot \lambda - 2) \cdot (A \cdot \lambda - 1) \cdot (A \cdot \lambda - 2)$$

$$= \ldots = \Gamma(A \cdot \lambda - \alpha) \cdot \prod_{k=1}^{\alpha} (A \cdot \lambda - k)$$

which gives a simple proof of (*) for this case. The more general case can be derived from [3], 11.1.3.1, p. 223.

Note also that for the variances, we get

$$Var(A \cdot X) = A^2 \cdot \frac{\alpha \cdot \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} = \frac{\alpha}{\lambda^2} \cdot \frac{A \cdot \lambda - \alpha}{A \cdot \lambda + 1} \to \frac{\alpha}{\lambda^2} = Var(Y)$$

for $A \to \infty$.

The following figures show some cases for different values of the parameters, the p.d.f. of X in red, the p.d.f. of Y in blue.

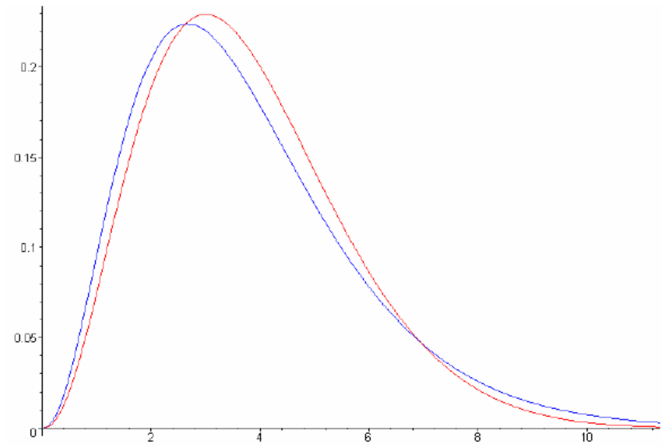

Figure 1. $A = 20,\ \alpha = 3.4,\ \lambda = 0.9,\ \beta = 14.6.$

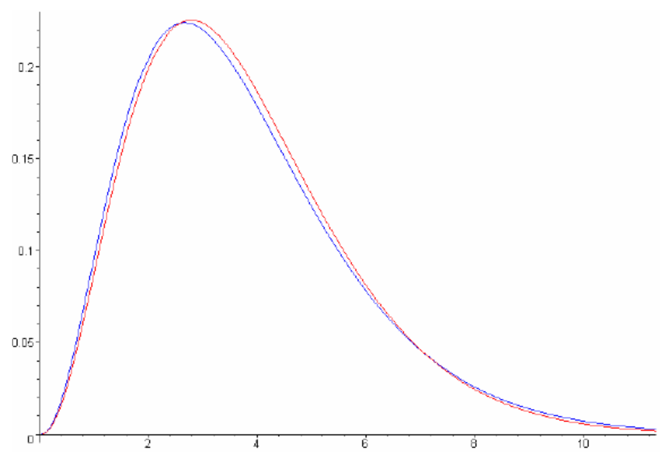

Figure 2. $A = 50,\ \alpha = 3.4,\ \lambda = 0.9,\ \beta = 41.6.$

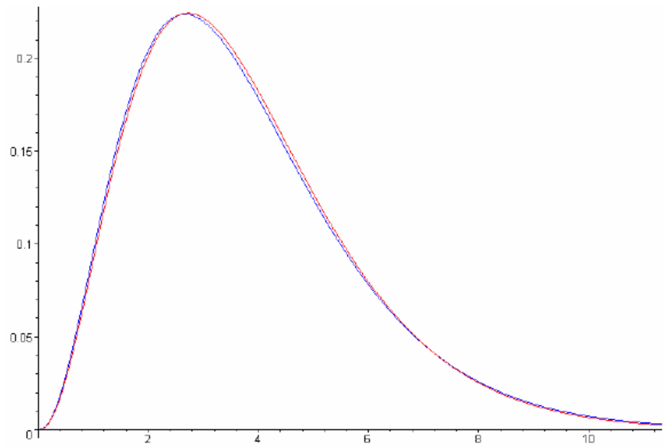

Figure 3. $A = 100, \; \alpha = 3.4, \; \lambda = 0.9, \; \beta = 86.6.$

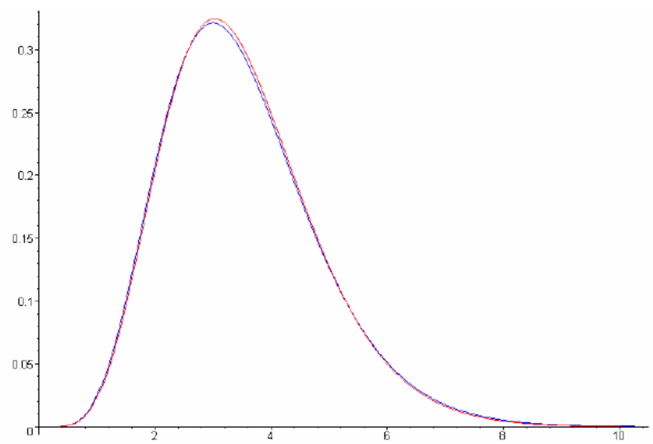

Figure 4. $A = 100, \; \alpha = 7, \; \lambda = 2, \; \beta = 193.$

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[3] A. Jeffrey, Handbook of Mathematical Formulas and Integrals, 2nd ed., Academic Press, London, 2000.