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NUMERICAL SOLUTION OF TWO DIMENSIONAL POISSON EQUATIONS BY USING FOURTH-ORDER COMPACT FINITE DIFFERENCE METHOD

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Abstract

In this paper, the fourth-order compact finite difference scheme has been presented for solving the two-dimensional Poisson equation. First, the given solution domain is discretized with uniform and non-uniform mesh size and then the partial derivative is replaced into functional values at each grid point by using Taylor series expansion. From this discretization, we obtain system of algebraic equations. Then, the obtained system of algebraic equations is solved by the Thomas method. The stability and convergent analysis of present scheme are investigated. To validate the applicability of the proposed method, one model example is considered and solved for different values of the mesh sizes in both directions. Numerical results are presented in tables in terms of root mean square error L_2 and maximum absolute error L_∞

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norms. The numerical results presented in tables and graphs confirm that the approximate solution is in good agreement with the exact solution.

1. Introduction

A partial differential equation (PDE) is an equation containing a partial derivative of the dependent variable [9]. These equations arise in almost all areas of applied mathematics, physics, and some branches of engineering [5], for instance, in fluid mechanics, elasticity, heat transfer, energy systems, environmental flows, hydraulics, neutron diffusion in nuclear reactors, and structural analysis [18]. This partial differential equation is classified into parabolic, hyperbolic, and elliptic types of equations [5, 9, 11, 16]. Therefore the partial differential equations model sorts of phenomena, display different behavior, and require different numerical techniques for their solution [11] and simplest examples of the elliptic type of PDEs are Poisson's equation and Laplace equation [5]. These elliptic-type equations are generally associated with equilibrium or steady-state problems [5]. For instance Steady-state condition in a communications circuit and electrical circuit are application of these types of equation.

The Steady-state condition in a communications circuit can be defined as a condition in which some specified characteristic of a condition, such as a value, rate, periodicity, or amplitude, exhibits only negligible change over an arbitrarily long period. Again the Steady-state conditions, in an electrical circuit define as the condition that exists after all initial transients or fluctuating conditions have damped out, and all currents, voltages, or fields remain essentially constant or oscillate uniformly [12]. For example, the velocity potential for the steady flow of incompressible non-viscous fluid satisfies Laplace's equation and the electric potential associated with a two-dimensional electron distribution of charge density satisfies Poisson's equation [5]. The Poisson equation is a generalization form of Laplace's equation. This equation is named after the French mathematician geometer, and physicist Simon Denis Poisson [1].

Boundary conation of Elliptic types of PDEs equation arises in the study of steady-state or time-independent solutions of heat equations. Because these solutions do not depend on time, initial conditions are irrelevant and only boundary conditions are specified. Applications of Poisson equation also include the static displacement $U(x, t)$ of a stretched membrane fastened in space along the boundary of a region; the electrostatic and gravitational potentials in certain force fields; and, in fluid mechanics for an ideal fluid [17].

Poisson's equation is also a very powerful tool for modeling the behavior of electrostatic systems, but unfortunately may not only be solved analytically for very simplified models [1]. Because of this, these methods are based on advanced mathematical techniques [5]. Among Elliptic types of PDEs, Poisson type equations are the most practical and frequently investigated [1]. In solving these types of partial differential equations, we are looking for a function of more than a variable that satisfies the same relation between different partial derivatives [11]. But Poisson equation may not only be solved analytically for very simplified models. Consequently, numerical simulation must be utilized for that model problem due to them has complex geometries behavior within their practical value [1]. Therefore, several numerical methods are available that we use to solve the Poisson equation. The numerical methods are, in general, simple but generate erroneous results [5].

In many application areas, such as aero-acoustics and electromagnetic, the propagation of acoustic and electromagnetic waves needs to be accurately simulated over very long periods and far distances. Some numerical methods are not accurate for solving such types of the equation. For instance, the finite difference method is used as the direct conversion of the partial differential equation from continuous function and operator into their discretely sampled counterpart. This converts the entire problem into a system of linear equations that may be readily solved employing matrix inversion, Jacobi, Gauss-elimination, the successive over-relaxation method [7]. The accuracy of such a method is therefore directly tied to the ability of a finite grid to approximate a continuous system and errors may be arbitrarily reduced by simply increasing the number of samples [6]. But the method has required high cost regarding storage capacity in the computational domain. Anley [16] solved the elliptic equation by using the finite volume method. He used the finite volume method and the solution domain is subdivided into a finite number of small control volumes by a grid that grid defines the boundaries of the control volumes while the computational node lies at the center of the control volume to solve the elliptic equation. Nodal points are used within these control volumes for interpolating the field variable and usually, the single node at the center of the control volume is used for each control volume. But the method gives better accuracy only for the small number of a grid point and is difficult to compute the solution in a complex computational domain when step length is very small. Genet and Lemi [1] presented the solution of two-dimensional Poisson equations using the finite difference method. This method is mathematically simple and guarantees the necessary accuracy for a relatively small number of mesh-size. This confirms that the method is not accurate for relatively a few grid points (i.e., for mesh size very large) and is difficult to apply for high dimension geometric spaces. Hence this method does not always converge to the exact solutions for coarser step lengths. Mohammad and Azim [20] also presented the Numerical Solution of Poisson's Equation Using a Combination of Logarithmic and Multiquadric Radial Basis Function Networks. In multiquadric radial basis functions MQ-RBFs, some parameters influence the accuracy of the solution. The solution diverges until the optimal shape parameters are obtained. As we compared to the exact solution, the approximate solution needs further improvement.

Even though the accuracy of the aforementioned methods is promising, they require large memory and long computational time. Besides, the methods are not suitable for higher-dimensional and problems involving complex geometries. So, the treatment of the mesh size and shape parameter in the applied method presents severe difficulties that have to be addressed to ensure the accuracy of the solution for the Poisson equation, and efficiency of the method applied. Therefore due to this end, the accumulation of errors is generated throughout solving the Poisson equation. Thus still, the accuracy of the method needs attention; because the treatment of the method used to solve the Poisson equation is not trivial distribution.

To reduce the accumulation of errors, the numerical algorithm must be highly accurate. To accomplish this goal, high-order compact finite difference schemes have been developed to solve PDEs types of the equation in a different application (see [8, 14, 19]). High-order finite difference schemes can be classified into two main categories: explicit schemes and Pade-type or compact schemes. Explicit schemes compute the numerical derivatives directly at each grid by using large stencils, while compact schemes obtain all the numerical derivatives along a grid line using smaller stencils and solving a linear system of equations. Experience has shown that compact schemes are much more accurate than the corresponding explicit scheme of the same order [14]. Therefore to this end, this paper aims to apply the fourth-order compact finite difference method that is capable of solving the Two-Dimensional Poisson Equation and obtain an innovative solution of Poisson Equation in the specified solution domain.

Statement of the problem

Consider that the following Poisson equation which is considered in [1] given by:

$$U_{xx} + U_{yy} = -f(x, y), \quad (x, y) \in (a, b) \times (c, d) \tag{1}$$

which is subject to Dirichlet boundary condition.

$$U(a, y) = U(b, y) = U(x, c) = U(x, d) = 0, \tag{2}$$

$f(x, y)$ is assumed to be sufficiently smooth functions in $D = [a, b] \times [c, d]$ for the existence and the uniqueness of the solution.

Discretized the solution domain

Now we define a mesh size h and k and the constant grid point by drawing a horizontal and vertical line of distance $'h'$ and $'k'$ respectively in the 'x' and 'y' directions. These lines are called gridlines and the points at which they interact are known as the mesh points. The mesh point that lies at end of the domain is called the boundary point. The solution to the problem that lies at boundary points is called boundary condition. These boundary conditions are used to find the solution of the given model problem at interior points. The points that lie inside the region (inside the solution domain) are called interiors points. The goal is to approximate the solution $'U_{jn}'$ at the interior mesh points. Hence we discretized the solution domain and generate a grid by using both uniform and non-uniform discretize of grid point given as follows.

* A uniform Cartesian grid point can be generated as:

$$a = x_0 < x_1 < x_2 < \dots < x_M = b, \quad x_{j+1} = x_j + jh, \quad h = \frac{b-a}{M};$$

$$c = y_0 < y_1 < y_2 < \dots < y_N = d, \quad y_{n+1} = y_n + nk, \quad k = \frac{d-c}{N}. \quad (3)$$

* A non-uniform Cartesian grid point can be generated as:

$$x_j = h \times rand(1), \quad h = x_{j+1} - x_j,$$

$$y_n = k \times rand(1), \quad k = y_{n+1} - y_n, \quad (4)$$

where $j = 0(1)M$, $n = 0(1)N$. M and N are the maximum numbers of grid points, respectively, in the x and y-direction. Then the present paper is organized as follows. Section two is a description of numerical methods, section three, stability and convergence analysis, section four is the results of numerical experiments, section five is the discussion and section six is the conclusion.

2. Formulation of the Numerical Scheme

Assuming that $U(x, y)$ has continuous higher order partial derivative on the region $D = [a, b] \times [c, d]$. For the sake of simplicity, we use $U(x_j, y_n) = U_{jn}$, $\frac{\partial^p U}{\partial x^p} = \partial_x^p U_{jn}$ and $\frac{\partial^p U}{\partial y^p} = \partial_y^p U_{jn}$ for $p \geq 1$ is p^{th} order derivatives. To construct the scheme, assume that for the approximate value of the following from the model problem in Eq. (1) as follow:

$$(U_{xx} + U_{yy})(x_0, y_0) \approx a_0 U_0 + a_1(U_1 + U_3) + a_2(U_2 + U_4) + a_3(U_5 + U_6 + U_7 + U_8), \tag{5}$$

$$f(x_0, y_0) \approx b_0 f_0 + b_1(f_1 + f_2 + f_3 + f_4). \tag{6}$$

By using Taylor series expansion, we have:

$$U_1 = U_0 + h\partial_x U_0 + \frac{h^2}{2!}\partial_x^2 U_0 + \frac{h^3}{3!}\partial_x^3 U_0 + \frac{h^4}{4!}\partial_x^4 U_0 + \frac{h5}{5!}\partial_x^5 U_0 + O(h^6), \tag{7}$$

$$U_3 = U_0 - h\partial_x U_0 + \frac{h^2}{2!}\partial_x^2 U_0 - \frac{h^3}{3!}\partial_x^3 U_0 + \frac{h^4}{4!}\partial_x^4 U_0 - \frac{h5}{5!}\partial_x^5 U_0 + O(h^6), \tag{8}$$

$$U_2 = U_0 + h\partial_y U_0 + \frac{h^2}{2!}\partial_y^2 U_0 + \frac{h^3}{3!}\partial_y^3 U_0 + \frac{h^4}{4!}\partial_y^4 U_0 + \frac{h5}{5!}\partial_y^5 U_0 + O(h^6), \tag{9}$$

$$U_4 = U_0 - h\partial_y U_0 + \frac{h^2}{2!}\partial_y^2 U_0 - \frac{h^3}{3!}\partial_y^3 U_0$$

$$+\frac{h^4}{4!}\partial_y^4 U_0 - \frac{h5}{5!}\partial_y^5 U_0 + O(h^6). \tag{10}$$

Adding Eq. (7) to Eq. (8) and Eq. (9) to Eq. (10), we obtain

$$a(U_1 + U_3) = a_1\left(2U_0 + h\partial_x U_0 + \frac{h^4}{4!}\partial_x^4 U_0\right) + O(h^6),$$

$$a(U_2 + U_4) = a_1\left(2U_0 + h\partial_y U_0 + \frac{h^4}{4!}\partial_y^4 U_0\right) + O(h^6), \tag{11}$$

where $T_1 = \frac{h^6}{360}\partial_x^6 U_0$ and $T_2 = \frac{h^6}{360}\partial_x^6 U_0$ are their local truncation errors. Again using the Taylor series expansion we have:

$$\begin{aligned} U_6 &= U_0 + h\left(\partial_x U_0 + \partial_y U_0\right) + \frac{h^2}{2!}(\partial_x^2 + 2\partial_{xy}^2 U_0 + U_0\partial_y^2 U_0) \\ &+ \frac{h^3}{3!}(\partial_x^3 U_0 + 3\partial_{xxy}^3 U_0 + 3\partial_{yyx}^3 U_0 + \partial_y^3 U_0) \\ &+ \frac{h^4}{4!}(\partial_x^4 U_0 + 4\partial_{xxxy}^4 U_0 + 6\partial_{xxyy}^4 U_0 + 4\partial_{xyyy}^4 U_0 + \partial_y^4 U_0) \\ &+ \frac{h^5}{5!}(\partial_x^5 U_0 + 5\partial_{x^4 y}^5 U_0 + 10\partial_{x^3 y^2}^5 U_0 + 10\partial_{x^2 y^3}^5 U_0 \\ &+ 5\partial_{xy^4}^5 U_0 + \partial_y^5 U_0) + O(h^6), \end{aligned} \tag{12}$$

$$\begin{aligned} U_5 &= U_0 - h\left(h\partial_x U_0 + \partial_y U_0\right) + \frac{h^2}{2!}(\partial_x^2 - 2\partial_{xy}^2 U_0 + U_0\partial_y^2 U_0) \\ &- \frac{h^3}{3!}(\partial_x^3 U_0 + 3\partial_{xxy}^3 U_0 + 3\partial_{yyx}^3 U_0 + \partial_y^3 U_0) \\ &+ \frac{h^4}{4!}(\partial_x^4 U_0 - 4\partial_{xxxy}^4 U_0 + 6\partial_{xxyy}^4 U_0 - 4\partial_{xyyy}^4 U_0 + \partial_y^4 U_0) \\ &- \frac{h^5}{5!}(\partial_x^5 U_0 + 5\partial_{x^4 y}^5 U_0 + 10\partial_{x^3 y^2}^5 U_0 + 10\partial_{x^2 y^3}^5 U_0 \end{aligned}$$

$$+ 5\partial^5_{xy^4}U_0 + \partial^5_y U_0) + O(h^6), \tag{13}$$

$$U_7 = U_0 + h(h\partial_x U_0 + \partial_y U_0) + \frac{h^2}{2!}(\partial^2_x + 2\partial^2_{xy}U_0 + U_0\partial^2_y U_0)$$

$$+ \frac{h^3}{3!}(\partial^3_x U_0 + 3\partial^3_{xxy}U_0 + 3\partial^3_{yyx}U_0 + \partial^3_y U_0)$$

$$+ \frac{h^4}{4!}(\partial^4_x U_0 + 4\partial^4_{xxxy}U_0 + 6\partial^4_{xxyy}U_0 + 4\partial^4_{xyyy}U_0 + \partial^4_y U_0)$$

$$+ \frac{h^5}{5!}(\partial^5_x U_0 + 5\partial^5_{x^4y}U_0 + 10\partial^5_{x^3y^2}U_0 + 10\partial^5_{x^2y^3}U_0$$

$$+ 5\partial^5_{xy^4}U_0 + \partial^5_y U_0) + O(h^6), \tag{14}$$

$$U_8 = U_0 - h(h\partial_x U_0 + \partial_y U_0) + \frac{h^2}{2!}(\partial^2_x - 2\partial^2_{xy}U_0 + U_0\partial^2_y U_0)$$

$$- \frac{h^3}{3!}(\partial^3_x U_0 + 3\partial^3_{xxy}U_0 + 3\partial^3_{yyx}U_0 + \partial^3_y U_0)$$

$$+ \frac{h^4}{4!}(\partial^4_x U_0 - 4\partial^4_{xxxy}U_0 + 6\partial^4_{xxyy}U_0 - 4\partial^4_{xyyy}U_0 + \partial^4_y U_0)$$

$$- \frac{h^5}{5!}(\partial^5_x U_0 + 5\partial^5_{x^4y}U_0 + 10\partial^5_{x^3y^2}U_0 + 10\partial^5_{x^2y^3}U_0$$

$$+ 5\partial^5_{xy^4}U_0 + \partial^5_y U_0) + O(h^6). \tag{15}$$

Now adding Eqs. (11-14) all together we obtain:

$$a_3(U_5 + U_6 + U_7 + U_8) = a_3[4U_0 + 2h^3(\partial^2_x U_0 + \partial^2_y U_0)$$

$$+ \frac{4h^4}{4!}(\partial^4_x U_0 + 6\partial^4_{xxyy}U_0 + \partial^4_y U_0) + O(h^6)]. \tag{16}$$

$T_3 = \frac{h^6}{360}(\partial^6_x + \partial^6_y)U_0$. Is it a local truncation error? From the model problem we have:

$$f_0 = -[\partial_x^2 U_0 + \partial_y^2 U_0],$$

$$\partial_x^2 f_0 = -[\partial_{x^4}^4 U_0 + \partial_{x^2 y^2}^4 U_0],$$

$$\partial_y^2 f_0 = -[\partial_{y^2 x^2}^4 U_0 + \partial_{y^4}^4 U_0]. \tag{17}$$

Using Eq. (17), from Eq. (6) we obtain:

$$b_0 f_0 = -b_0[\partial_x^2 U_0 + \partial_y^2 U_0], \tag{18}$$

$$b_1(f_1 + f_2 + f_3 + f_4) = b_1\left[4f_0 + \frac{2h^2}{2}(\partial_x^2 f_0 + \partial_y^2 f_0) + O(h^4)\right]$$

$$= -b_1\left[4(\partial_x^2 U_0 + \partial_y^2 U_0) + \frac{2h^2}{2}(\partial_{x^4}^4 U_0 + \partial_{x^2 y^2}^4 U_0 + \partial_{y^2 x^2}^4 U_0 + \partial_{y^4}^4 U_0\right]$$

$$+ O(h^4). \tag{19}$$

Now substituting Eqs. (5) and (6) into the model problem in Eq. (1), we obtain:

$$a_0 U_0 + a_1(U_1 + U_3) + a_2(U_2 + U_4) + a_3(U_5 + U_6 + U_7 + U_8)$$

$$= b_0 f_0 + b_1(f_1 + f_2 + f_3 + f_4). \tag{20}$$

Again substituting Eqs. (11), (16), (18) and (19) into Eq. (20), we obtain:

$$a_0 U_0 + a_1\left(2U_0 + h\partial_x U_0 + \frac{h^4}{4!}\partial_x^4 U_0\right) + a_1\left(2U_0 + h\partial_y U_0 + \frac{h^4}{4!}\partial_y^4 U_0\right)$$

$$+ a_3\left[4U_0 + 2h^3(\partial_x^2 U_0 + \partial_y^2 U_0) + \frac{4h^4}{4!}(\partial_x^4 U_0 + 6\partial_{xxyy}^4 U_0 + \partial_y^4 U_0)\right]$$

$$= -b_0[\partial_x^2 U_0 + \partial_y^2 U_0]$$

$$- b_1 \left[4(\partial_x^2 U_0 + \partial_y^2 U_0) + \frac{2h^2}{2} (\partial^4_{x^4} U_0 + \partial^4_{x^2 y^2} U_0 + \partial^4_{y^2 x^2} U_0 + \partial^4_{y^4} U_0 \right].$$

This gives the system of linear equation in the form of:

$$a_0 + 2a_1 + 2a_2 + 4a_4 = 0,$$

$$h^2(a_1 + 2a_3) = -b_0 - 4b_1,$$

$$h^2(a_2 + 2a_3) = -b_0 - 4b_1,$$

$$h^2 a_3 = -2b_1,$$

$$h^2(a_1 + 2a_3) = -12b_1,$$

$$h^2(a_2 + 2a_3) = -12b_1. \tag{21}$$

Now by solving the system of linear equation in Eq. (21), we obtain the value of arbitrary constant given by: $a_1 = a_2 = -\frac{8b_1}{h^2}$, $a_3 = -\frac{2b_1}{h^2}$, $a_0 = 40\frac{b_1}{h^2}$, $b_0 = 8b_1$.

Now after certain simplification with $b_1 = 1$, in Eq. (20), we obtain the proposed scheme given by

$$4[U_1 + U_2 + U_3 + U_4] + [U_5 + U_6 + U_7 + U_8] - 20U_0$$

$$= -\frac{h^2}{2}(8f_0 + f_1 + f_2 + f_3 + f_4).$$

Implies that:

$$4[U_{i+1,\,j} + U_{i,\,j+1} + U_{i-1,\,j} + U_{i,\,j-1}]$$

$$+ [U_{i+1,\,j-1} + U_{i+1,\,j+1} + U_{i-1,\,j+1} + U_{i-1,\,j-1}] - 20U_{i,\,j}$$

$$= -\frac{h^2}{2}\left(8f_{i,j} + f_{i,j+1} + f_{i,j+1} + f_{i-1,j} + f_{i,j-1}\right). \tag{22}$$

With its local truncation is:

$$T_{i,j} = 12h^4\left(\frac{(\Delta y)^4}{360}\partial_y^6 - \frac{(\Delta x)^4}{360}\partial_x^4 + \frac{(\Delta y)^2}{12}\partial_y^3 - \frac{(\Delta x)^2}{12}\partial_y^2\right)U_{i,j}. \tag{23}$$

Hence from Eq. (22), we obtain tri-diagonal coefficient matrix of system of linear equation. To solve this system of equation, we use the Thomas method. Because of to solve these types of system of the equation the most recommended numerical method is the Thomas method. This is due to the coefficient matrix contains several zero entries.

3. Stability Analysis and Convergent of the Proposed Method

The Fourier analysis (Von-Neumann) stability analysis technique is applied to investigate the stability analysis of the proposed method. Such an approach has been used by many researchers like [3, 4, 13, 21, 22]. Now assume that the trial solution of the given problem at the points (x_i, y_j) is

$$u_{jn} = \lambda^j e^{piK_a}, \tag{24}$$

where $p = \sqrt{-1}$, $K_a = a\pi/N$, $k \in$ ($\mathbb{R}$ set of a real number), $\lambda \in$ (set of a complex number) and $a = 1(1)N$. Substituting Eq. (24) into Eq. (22), we obtain:

$$4[\lambda^j e^{phK_a(i+1)} + \lambda^{j+1}e^{pihK_a} + \lambda^j e^{phK_a(i-1)} + \lambda^{j-1}e^{pihK_a}]$$

$$+ [\lambda^{j-1}e^{phK_a(i+1)} + \lambda^{j+1}e^{phK_a(i+1)} + \lambda^{j+1}e^{phK_a(i-1)}$$

$$+ \lambda^{j-1}e^{pihK_a(i-1)}] - 20\lambda^j e^{pihK_a}$$

$$= -\frac{h^2}{2}(8\lambda^j e^{pihK_a} + \lambda^{j+1}e^{pihK_a} + \lambda^{j-1}e^{phK_a(i+1)}$$

$$+ \lambda^j e^{phK_a(i-1)} + \lambda^{j+1} e^{phK_a(i-1)}).$$

On dividing both sides of this equation by $\lambda^j e^{pihK_a}$, we obtain:

$$4[e^{phK_a} + \lambda_a + e^{-phK_a} + \lambda_a^{-1}] + [\lambda_a^{-1} e^{phK_a} + \lambda_a e^{phK_a}$$

$$+ \lambda_a e^{-phK_a} + \lambda_a^{-1} e^{-phK_a}] - 20$$

$$= -\frac{h^2}{2}(8 + \lambda_a + e^{phK_a} + e^{-phK_a} + \lambda_a^{-1}),$$

$$16\cos(hk_a) + 8\lambda_a + 8\lambda_a^{-1} + 4\lambda_a^{-1}\cos(hk_a) + 4\lambda_a\cos(hk_a) - 40$$

$$= 8h^2 + h^2\lambda_a + 2h^2\cos(hk_a) + h^2\lambda_a^{-1}.$$

This implies that:

$$\lambda_a\left(2 + \cos(hk_a) - \frac{1}{4}h^2\right) + \lambda_a^{-1}\left(2 + \cos(hk_a) - \frac{1}{4}h^2\right)$$

$$+\left[\left(4 - \frac{1}{2}h^2\right)\cos(hk_a) - 2(5 + h^2)\right] = 0.$$

Multiplying both sides of the above equation by λ_a, we obtain:

$$\lambda_a^2 - \lambda_a \frac{2[(8 - h^2)\cos(hk_a) - 4(5 + h^2)]}{(8 - 4\cos(hk_a) + h^2)} - \frac{(8 - 4\cos(hk_a) + h^2)}{(8 - 4\cos(hk_a) + h^2)} = 0,$$

$$\lambda_a^2 - \lambda_a \frac{2[(8 - h^2)\cos(hk_a) - 4(5 + h^2)]}{(8 + 4\cos(hk_a) + h^2)} - 1 = 0.$$

Let $X = [(8 - h^2)\cos(hk_a) - 4(5 + h^2)]$ and $Y = (8 + 4\cos(hk_a) + h^2)$.

$$\lambda_a^2 - \frac{2\lambda_a X}{Y} = 1.$$

By using perfect square, we have:

$$\left(\lambda_a - \frac{X}{Y}\right)^2 = 1 + \left(\frac{X}{Y}\right)^2.$$

$$\lambda_a = \frac{X}{Y} \pm \sqrt{1 + \left(\frac{X}{Y}\right)^2} = \frac{1}{Y}\,(X + \sqrt{(Y^2 + X^2)}). \tag{25}$$

Since for any value of mesh-size h, $|X| = |[(8 - h^2)\cos(hk_a) - 4(5 + h^2)]| \leq 1$ and $|Y| = |(8 + 4\cos(hk_a) + h^2)| > 1$. Hence from Eq. (25) we have:

$$|\lambda_a| = \left|\frac{X}{Y} \pm \sqrt{1 + \left(\frac{X}{Y}\right)^2}\right| = \left|\frac{1}{Y}\,(X + \sqrt{(Y^2 + X^2)}\right|$$

$$\leq \left|\frac{1}{Y}\right|\left|X \pm \sqrt{Y^2 + X^2}\right|$$

$$< \left|\frac{1}{Y}\right| |X| + \sqrt{|Y^2| + |X^2|} \quad \text{Triangular inequality}$$

$$\leq \left|\frac{1}{Y}\right| |X| + \sqrt{|Y|^2 + |X|^2} < 1.$$

Hence we obtain the required criteria for stability investigation of the proposed method. Therefore the proposed method is strictly stable for solving two-dimensional Poisson equations.

Theorem 2. *The difference equation given in the form of Eq.* (12) *is stable if for which the eigenvalues of the coefficient matrix of the system of the differential equation are satisfied Real* $(\lambda_j) < 0$.

Proof. See reference [3].

Since from the principal part of the local truncation error, the derived local truncation error for the proposed scheme is

$$T_{i,j} = \frac{h^2}{360}\,(2\partial_x^6 + 9\partial_{xxxy}^6 + 14\partial_{xxyy}^6 U_0 + 9\partial_{xyyy}^6 + 2\partial_y^6)U_{i,j}.$$

$$\lim_{h\to 0}\| T_{i,j} \| = \lim_{h\to 0} h^6 \| \frac{1}{360}(2\partial_x^6 + 9\partial_{xxxy}^6$$

$$+ 14\partial_{xxyy}^6 U_0 + 9\partial_{xyyy}^6 + 2\partial_y^6)U_{i,j} \| \to 0.$$

Thus this implies that, $T_{i,j} \to 0$ as $h \to 0$. So that, the scheme is consistent with the order of $O(\Delta x^6 + \Delta y^6) = O(h^6)$. Hence the scheme is convergent.

Criteria for investigating the accuracy of the method

This section presented the criteria that the accuracy of the present method is investigated. The accuracy of the solution will depend on how small we make the step size, $\Delta x = \Delta y = h$. To test the performance of the proposed method to give an accurate solution for the given model problem, maximum absolute error, L_2 and L_∞ norms are calculated by using the following formula:

$$L_\infty = \max_{1\le n\le N} | u(x_i, y_j) - u_{i,j} |,$$

$$L_2 = \sqrt{\frac{1}{N}\sum_{j=0}^{N} | u(x_i, y_j) - u_{i,j} |^2} \quad i = 1(1)M,$$

where N is the maximum number of step, $u(x_i, y_j)$ is the exact solution and $u_{i,j}$ is approximation solution of the Poisson equation in Eq. (1) at the grid point (x_i, y_j).

4. Numerical Experiments and their Results

To test the validity of the proposed method, we have considered the following three model problem considered in [1]. Numerical results and errors are computed and the outcomes are represented tabularly and graphically.

Example 1. Consider the classical two-dimensional equation considered in [1]

$$u_{xx} + u_{yy} = -2\pi^2 \sin(\pi x)\sin(\pi y), \quad (x, y) \in (0, 1) \times (0, 1).$$

The subjected Dirichlet boundary condition is given by:

$$U(0, y) = u(1, y) = u(x, 0) = u(x, y) = 0.$$

$\Delta x = \Delta y = h = \frac{1}{40}$ and the exact solution is given by:

$$u(x, y) = \sin(\pi x)\sin(\pi y).$$

Table 1. Comparison of Point-wise maximum absolute error (L_∞) and root mean square error (L_2) with uniform mesh size equal $\Delta x = \Delta y = h = 1/40$

Specific grid points		Point-wise maximum absolute error obtained by Genet Mekonnen and Lemi Guta in [1]	Point-wise maximum absolute error and root mean square error by present methods	
x	y	L_∞	L_∞	L_2
1/4	1/4	$2.65E-02$	$2.5710E-04$	$4.0651E-05$
1/2	1/4	$3.75E-02$	$3.6359E-04$	$5.7489E-05$
3/4	1/4	$2.65E-02$	$2.5710E-04$	$4.0651E-05$
1/4	1/2	$3.75E-02$	$3.6359E-04$	$5.7489E-05$
1/2	1/2	$5.30E-02$	$5.1420E-04$	$8.1302E-05$
3/4	1/2	$3.75E-02$	$3.6359E-04$	$5.7489E-05$
1/4	3/4	$2.65E-02$	$2.5710E-04$	$4.0651E-05$
1/2	3/4	$3.75E-02$	$3.6359E-04$	$5.7489E-05$
3/4	3/4	$2.65E-02$	$3.6359E-04$	$2.5710E-05$

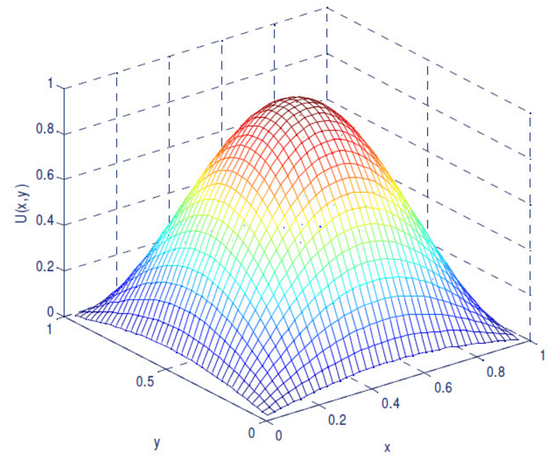

Figure 1. Physical Behavior of Approximate solution for given example on uniform mesh size $\Delta x = \Delta y = h = 1/40$.

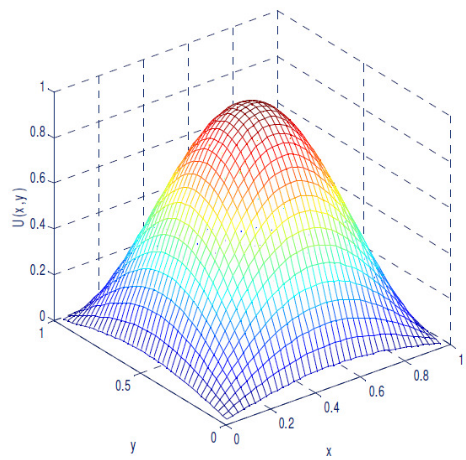

Figure 2. Physical Behavior of Exact solution for given example on uniform mesh size $\Delta x = \Delta y = h = 1/40$.

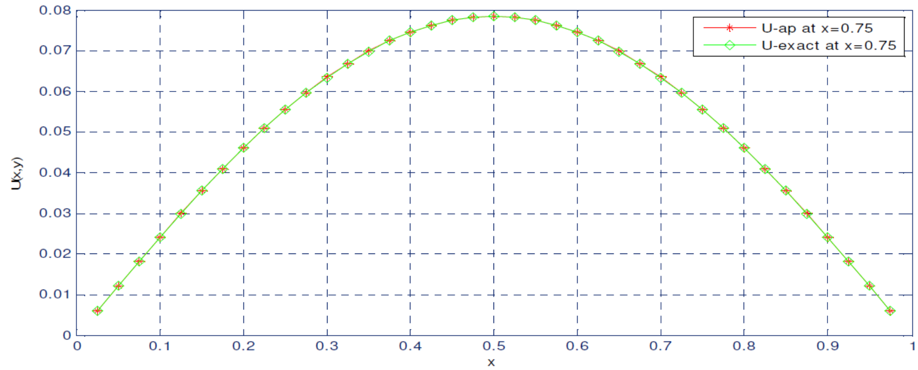

Figure 3. Variation of exact versus numerical solution for given example with uniform mesh size $\Delta x = \Delta y = h = 1/40$.

Table 2. Comparison of Point-wise maximum absolute error (L_∞) and root mean square error (L_2) with non-uniform mesh size equal

Specific grid points		Point-wise maximum absolute error obtained by Genet Mekonnen and Lemi Guta in [1]	Point-wise maximum absolute error and root mean square error by present methods	
x	y	L_∞	L_∞	L_2
0.127	0.0975	$3.51E-02$	$4.3477E-04$	$1.0248E-04$
0.6324	0.0975	$6.92E-02$	$2.0690E-03$	$4.8766E-04$
0.8147	0.0975	$1.015E-01$	$1.5356E-03$	$3.6195E-04$
0.127	0.2285	$9.16E-02$	$2.2017E-03$	$5.1895E-04$
0.6324	0.2785	$3.532E-01$	$1.6866E-03$	$3.9754E-04$
0.8147	0.2785	$1.269E-01$	$6.6610E-04$	$1.5700E-04$
0.127	0.5469	$9.59E-02$	$2.2450E-03$	$5.2914E-04$
0.6324	0.5469	$2.105E-01$	$1.1009E-03$	$2.5948E-04$
0.8147	0.5469	$7.33E-02$	$1.4153E-03$	$3.3358E-04$

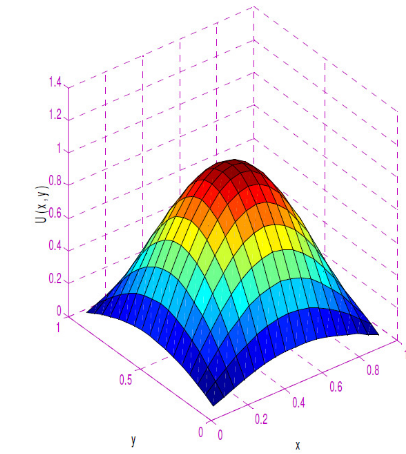

Figure 4. Physical Behavior of Approximate solution for given example on non-uniform mesh size.

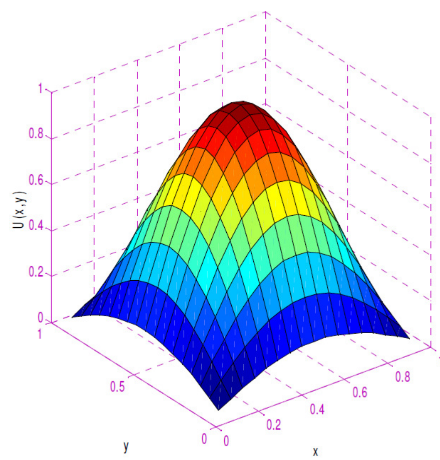

Figure 5. Physical Behavior of Exact solution for given example on non-uniform mesh size.

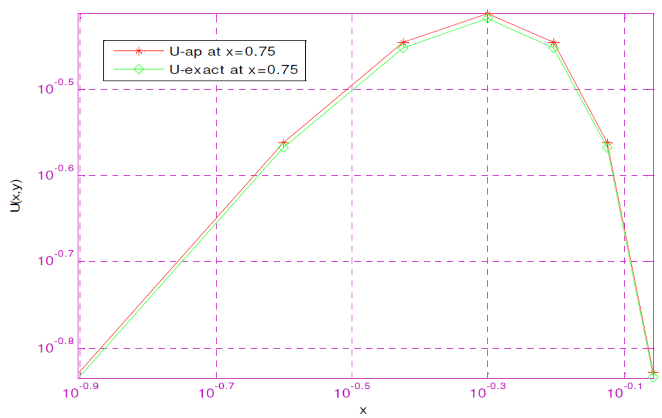

Figure 6. Variation of exact versus numerical solution for given example on non-uniform mesh size.

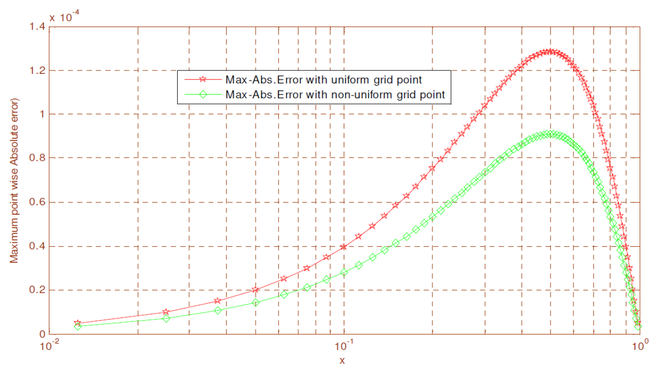

Figure 7. Variation of Point-wise absolute errors between exact and numerical solution for given example on uniform versus non-uniform mesh size.

5. Discussions

In this paper, we presented a fourth-order compact finite difference method to obtain an innovative solution for two-dimensional Poisson equations. The innovative solution, obtained within the fourth-order compact finite difference method, discussed only the case of the Dirichlet boundary condition. Regarding this partial differential equation, we note that there are two main ways of compact finite difference discretization (over a uniform grid and non-uniform grid points). When we apply the compact finite difference method to the continuous two-dimensional Poisson equation the equation is replaced by a "discrete" approximation. The number of those discrete points can be selected uniformly or non-uniformly depending on the mesh size (h). The mesh is the set of locations where the discrete solution is computed. Two key parameters of the mesh are the local distance between adjacent points in space. Fourth-order Compact finite difference discretization is simple to implement by using both equal mesh size and non-uniform mesh size as shown above in the table and graph. The full discretization of the Poisson equation by the present method leads to the system of linear equations which is solved by using the Thomas method. The convergence has been shown in the sense of maximum point-wise absolute error norm (L_∞) and root mean error (L_2), their values are given in tables and they are compared with pre-existing results. The stability and convergence of the present method are also investigated by using the Von-Neumann stability analysis technique. The results presented in Tables 1 and 2 demonstrate fourth-order finite difference method gives a more accurate numerical solution than the pre-existing method in the literature. As we see from Figure 7 the present method is more accurate when we investigate the solution of the model problem on non-uniform grid point discretization of the solution domain. Moreover, Figures 3 and 6 specifies that the present method gives an accurate solution for the 2D Poisson equation on both uniform and non-uniform grid point discretization of solution domain and the approximate exact solution very well.

6. Conclusion

The key purpose of this work is to formulate and investigate the fourth-order compact finite difference method for solving two-dimensional Poisson equations. To further collaborate the applicability of the proposed method; tables of point-wise absolute error and root mean square error and graphs have been plotted for Example 1, for the exact solution versus the numerical solutions at different values of x on both uniform and non-uniform grid points. Table 1, shows the absolute errors obtained by my fourth-order compact finite difference method have been compared with absolute errors obtained by [1] on uniform grid points and it shows that the present method is the more convergent method. Table 2, also shows the absolute errors obtained by the present method have been compared with absolute errors obtained by [1] on non-uniform grid points and then also it is showing that the present method is accurate than the previous method. Generally, the present method is computational: stable, effective, simple to use, convergent, and gives an accurate solution than some previously existing methods.

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