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NORM INEQUALITIES ON RIESZ POTENTIAL OPERATORS IN VARIABLE EXPONENT FOFANA SPACES

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Abstract

We introduce a class of Banach spaces $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$ (where $q()$, $p()$, $\alpha()$ are functions on Ω) which are subspaces of the two-variable exponent amalgam spaces $(L^{q()}, l^{p()})(\Omega)$. We exhibit some properties of these spaces and study the behavior of the Riesz potential operators on these spaces

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1. Introduction

Our purpose is to derive sufficient conditions for fractional integral operators (also known as Riesz potential) in three-variable exponent amalgam spaces (t-VEAS) $(L^{q()}, l^{p()})^{\alpha()}$ under the log-Hölder continuity condition on the exponent $q()$. The derived results are new even for constants q, p, α in the case of potential operators defined on $\mathbb{R}^d$.

The boundedness for fractional integral operators in variable exponent Lebesgue spaces was investigated by many authors: [1-4]. Apart from interesting theoretical considerations, the study of variable exponent spaces was motivated by a proposed application to modeling electrorheological fluids (see [5]), to image restoration (see, e.g., [6]), etc.

The amalgam spaces $(L^q, l^p)(\Omega)$ (with q, p constants) have been used by Wiener (see [23]) in connection with Tauberian theorems. Long after, Holland undertook their systematic study, (see [24]).

Since then, they have been extensively studied (see the survey paper [25] and the references therein) and generalized to locally compact groups (see [26-28]).

They may be looked at as spaces of functions that behave locally as elements of $L^q(\mathbb{R}^d)$ and globally as belonging to $l^p(\mathbb{Z}^d)$. Taking this into account, Feichtinger has introduced Banach spaces whose elements belong locally to some Banach space, and globally to another (see [29]).

The spaces $(L^q, l^p)^\alpha(\mathbb{R}^d)$ with constant exponents, have been introduced during 1988 by Ibrahim Fofana, see [7, 9]. We will always refer to them as Fofana spaces. It is well known that if $\alpha \in \{q, p\}$, then $(L^q, l^p)^\alpha(\mathbb{R}^d)$ coincides with the Lebesgue spaces L^α and for $q < \alpha$ the space $(L^q, l^\infty)^\alpha(\mathbb{R}^d)$ is exactly the classical Morrey spaces. Thus Fofana spaces can be viewed as some generalized spaces. Many classical results in Fourier analysis, well-known and widely used in Lebesgue or Morrey spaces, have been extended to the setting of Fofana spaces (see [7, 8, 9, 10, 11, 12, 13, 16, 17, 51, 52, 53]). Let us recall the definition of the spaces $(L^q, l^p)^\alpha(\mathbb{R}^d)$ with constant exponents.

Let $q,\ p,\ \alpha \in [1, \infty]$ and $0 < r < \infty$, we let:

(a) $$I_k^r = \prod_{j=1}^{d} [k_j r,\ (k_j + 1)r), \quad k = (k_j)_{1 \le j \le d} \in \mathbb{Z}^d,$$

$$J_x^r = \prod_{j=1}^{d} \left[x_j - \frac{r}{2},\ x_j + \frac{r}{2}\right), \quad x = (x_j)_{1 \le j \le d} \in \mathbb{R}^d,$$

$$Q(x,\ r) = \prod_{j=1}^{d} \left[x_j - \frac{r}{2},\ x_j + \frac{r}{2}\right], \quad x = (x_j)_{1 \le j \le d} \in \mathbb{R}^d.$$

For any subset E of $\mathbb{R}^d$, χ_E is the characteristic function of E and $|E|$ the Leesgue measure of E.

(b) For any $f \in L_{loc}^q(\mathbb{R}^d)$ we let:

$$_r\| f \|_{q,p} = \begin{cases} \left[\sum_{k \in \mathbb{Z}^d} \| f\chi_{I_k^r} \|_q^p\right]^{\frac{1}{p}} & \text{if} \quad p < \infty, \\ \sup_{k \in \mathbb{Z}^d} \| f\chi_{I_k^r} \|_q & \text{if} \quad p = \infty, \end{cases} \tag{1}$$

$$\| f \|_{q,p,\alpha} = \sup_{r>0} r^{d\left(\frac{1}{\alpha} - \frac{1}{q}\right)} {}_r\| f \|_{q,p}.$$

The amalgam spaces $(L^q, l^p)(\mathbb{R}^d)$ and $(L^q, l^p)^\alpha(\mathbb{R}^d)$ with constant exponents $q,\ p,\ \alpha$ are defined as follows:

$$(L^q, l^p)(\mathbb{R}^d) = \{f \in L_{loc}^q(\mathbb{R}^d) : {}_1\| f \|_{q,p} < \infty\},$$

$$(L^q, l^p)^\alpha(\mathbb{R}^d) = \{f \in L_{loc}^q(\mathbb{R}^d) : \| f \|_{q,p,\alpha} < \infty\}.$$

Recently One-variable exponent amalgam space $(L^{q()}, l^p)$ (where

$q()$ is a function, p is a constant belonging to $[1, \infty])$ has been widely studied in [18-21].

In 2023 [49], Yang and Zhou introduced the one-variable Fofana's spaces $(L^{p()}, l^q)^{\alpha}(\mathbb{R}^d)$ where $p()$ is a function such that $1 < p() < \infty$ and q, α are constant reals verifying $1 \leq q, \ \alpha \leq \infty,$ in this work some properties are derived and the pre-dual of those spaces were established which contributed to prove the necessary conditions of fractional integral commutators' boundedness. As applications, the characterization of fractional integral operators and commutators on (one-) variable Fofana's spaces are discussed, which are new result even for the classical Fofana's spaces.

In the recent past, N. Diarra [50] investigated the boundedness of classical operators such as Riesz potentials operators, maximal operators, Calderon-Zygmund operators and some generalized sublinear operators in both $(L^{p()}, l^q)^{\alpha}(\mathbb{R}^d)$ and their preduals $H(p'(), q', \alpha')(\mathbb{R}^d),$ in this paper some properties of these spaces are proved. The results extend and/or improve those of classical Fofana's spaces and their preduals.

The present work considers all exponents q, p, α of $(L^q, l^p)^{\alpha}(\mathbb{R}^d)$ as functions and generalizes the two Banach function spaces of the two papers of Diarra [50] and Yang and Zhou [49].

The following contributors have done huge work that contributed to the success of our paper: [22, 15, 47, 48].

In this work, we will extend the definition of these constant exponent spaces $(L^q, l^p)(\mathbb{R}^d),$ $(L^q, l^p)^{\alpha}(\mathbb{R}^d),$ $(q, p, \alpha$ are constants) to the case where $q, \ p, \ \alpha$ are functions $q(), \ p(), \ \alpha()$ from $\Omega \subset \mathbb{R}^d$ to $\mathbb{R}$.

It seems that Wiener amalgams with two or three variable exponents have not yet been considered in full generality.

In a precedent work we have defined and studied a two-variable exponent amalgam spaces $(L^{q()}, l^{p()})(\Omega);$ (where $q(), \ p()$ are functions, $\Omega \subset \mathbb{R}^d),$ see [22].

It is clear that $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$ is a subspace of the two-variable amalgam spaces $L^{q()}, l^{p()})(\Omega)$. We denote by $L^0 = L^0(\mathbb{R}^d) = L^0(\mathbb{R}^d, dx)$ the real vector space of equivalent classes (modulo equality Lebesgue almost everywhere) of Lebesgue measurable complex-valued functions on $\mathbb{R}^d$.

One of the most significant operators in Harmonic Analysis is the fractional integral operator I_γ, $(0 < \gamma < 1)$, also known as the Riesz potential (of order $d \cdot \gamma$) and defined by

$$I_\gamma f(x) = \int_{\mathbb{R}^d} | x - y |^{d(\gamma-1)} f(y) dy$$

for such $x \in \mathbb{R}^d$, $f \in L^0$ for which the above integral makes sense.

The boundedness properties of I_γ between various Banach spaces have been extensively studied, see [30-36]. Lately, the boundedness of the Riesz potential operator on variable exponent spaces was also studied by many researchers, see [37-44].

In this work, we define the variable exponent Fofana's spaces $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$ and give some properties and study the continuity of the Riesz potential operators.

The paper is divided into four sections. Section 2 includes fundamental notations and definitions which will be used in the subsequent sections. Section 3 contains auxiliary results and properties. Section 4 deals with the behavior of the Riesz potential on $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$.

Throughout the paper, the constants are independent of the main parameters involved, with values which may different from line to line.

2. Definitions and Notations

Notation 1.

- d will be a fixed positive integer, Ω a non void subset of $\mathbb{R}^d$, the

d-dimensional Euclidean space $\mathbb{R}^d$ is equipped with the usual Euclidean norm $|x|$.

$p(), q(), r(), \ldots$ in general indicate that $p, q, r, \ldots$ are functions used as norm indexes $(\|\cdot\|_{p()}, \|\cdot\|_{q()}, \|\cdot\|_{r()}, \ldots)$.

$f(\cdot), g(\cdot), h(\cdot), \ldots$ in general mean that $f, g, h, \ldots$ are functions which are applied on the elements $x, y, z, \ldots$ of $\mathbb{R}^d$, the dots between the brace refer to these elements.

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p() : \Omega \to [1, \infty]$. In order to distinguish between variable and constant exponents, we will always denote exponent functions by $p()$.

Given $p()$ and a set $E \subset \Omega$, let:

$$p_-(E) = \text{ess}\inf_{x\in E} p(x), \quad p_+(x) = \text{ess}\sup_{x\in E} p(x).$$

We simply write:

$$p_- = p_-(\Omega) \quad \text{and} \quad p_+ = p_+(\Omega).$$

As in the case for the classical Lebesgue spaces, we will encounter different behavior depending on whether $p(x) = 1$, $1 < p(x) < \infty$, $p(x) = \infty$. Therefore, we define three canonical subsets of Ω:

$$\Omega_\infty^{p()} = \Omega_\infty = \{x \in \Omega : p(x) = \infty\},$$

$$\Omega_1^{p()} = \Omega_1 = \{x \in \Omega : p(x) = 1\},$$

$$\Omega_*^{p()} = \Omega_* = \{x \in \Omega : 1 < p(x) < \infty\}.$$

Below, the value of certain constants will depend on whether these sets have positive measure; if they do we will use the fact that, for instance

$$\left\| \chi_{\Omega_1^{p(\cdot)}} \right\|_\infty = 1.$$

Given $p()$, we define the conjugate exponent $p'()$ by:

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \Omega$$

with the convention $\frac{1}{\infty} = 0$.

Since $p()$ is a function, the notation $p'()$ can be mistaken for the derivative of $p()$, but we will never use the symbol <<′>> in this sense.

The notation p' will always denote the conjugate of a constant exponent. The operation of taking the supremum/infimum of an exponent does not commute with forming the conjugate exponent. In fact, a straightforward computation shows that:

$$(p'())_+ = (p_-)', \quad (p'())_- = (p_+)'.$$

For simplicity, we will omit one set of parentheses and write the left-hand side of each equality as:

$$p'()_+ = (p_-)', \quad p'()_- = (p_+)'.$$

A function $r() : \Omega \to \mathbb{R}$ is locally log-Hölder continuous and denote this by $r() \in LH_0(\Omega)$, if there exists a constant C_0 such that

$$| x - y | \le \frac{1}{2} : | r(x) - r(y) | \le \frac{C_0}{-\log(| x - y |)}, \quad x, y \in \Omega.$$

We say that $r()$ is log-Hölder continuous at infinity and denote this by $r() \in LH_\infty(\Omega)$, if there exist C_∞ and $r_\infty = r(\infty)$ such that

$$| r(x) - r_\infty | \le \frac{C_\infty}{\log(e + | x |)}, \quad x \in \Omega.$$

If $r()$ is log-Hölder continuous locally and at infinity, we will denote this by writing $r() \in LH(\Omega) = LH_0(\Omega) \cap LH_\infty(\Omega)$.

If there is no confusion about the domain, we will sometimes write: LH_0, LH_∞ or LH.

• In the whole document $\mathbb{N} = \{1, 2, ...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1.

1. For any $q() \in \mathcal{P}(\Omega)$ and a Lebesgue measurable function f, we denote $\rho_{L^{q()}(\Omega)}(f)$ by:

$$\rho_{L^{q()}(\Omega)}(f) = \int_{\Omega \setminus \Omega_\infty^{q()}} | f(x) |^{q(x)} dx + \| f \|_{L^\infty(\Omega_\infty^{q()})}, \tag{2}$$

where

$$\| f \|_{L^\infty(\Omega_\infty^{q()})} = \inf\{\varepsilon > 0 : | f(x) | \leq \varepsilon \quad a.e. \quad x \in \Omega_\infty^{q()}\}.$$

We define

$$\| f \|_{L^{q()}(\Omega)} = \| f \|_{q(), \Omega} = \inf\left\{\lambda > 0 : \rho_{q()}\left(\frac{f}{\lambda}\right) \leq 1\right\}, \tag{3}$$

$$L^{q()}(\Omega) = \{f \in L^0(\Omega) : \| f \|_{L^{q()}(\Omega)} < \infty\}. \tag{4}$$

2. If f is unbounded on $\Omega_\infty^{q()}$ or $f()^{q()} \notin L^1(\Omega^{q()} \setminus \Omega_\infty)$, we define $\rho_{q()}(f) = \infty$.

3. If $q_+ < \infty$, in particular when $| \Omega_\infty^{q()} | = 0$, we let $\| f \|_{L^\infty(\Omega_\infty^{q()})} = 0$.

4. If $q_+ = \infty$, then $\inf\{\varepsilon > 0 : q(x) \leq \varepsilon \; a.e. \; x \in \Omega\} = \infty$, that is, $\forall \varepsilon > 0 : q(x) > 0 \; a.e. \; x \in \Omega$, therefore in this case $q()$ is unbounded,

$$\Omega_\infty^{q()} = \{x \in \Omega : q(x) = \infty\} = \Omega, \quad \text{and} \quad \rho_{q(), \Omega}(f) = \sup_{x \in \Omega} | f(x) |.$$

5. If $| \Omega \setminus \Omega_\infty^{q()} | = 0$, then $\rho_{q()}(f) = \| f \|_{L^\infty(\Omega_\infty^{q()})}$.

Definition 2. Let I be a non void countable set, $\mathcal{P}(I)$ be the set of all Lebesgue measurable functions $p() : I \to [1, \infty]$, $\mathcal{P}(I) = \{p() / p() : I \to [1, \infty], \; p() \text{ is Lebesgue measurable}\}$.

1. For any $p() \in \mathcal{P}(I)$ and $\{a_k\}_{k \in I} \in \mathbb{R}^I$, we define the modular $\rho_{l^{p()}(I)}$ by:

$$\rho_{l^{p()}(I)}(\{a_k\}_{k \in I}) = \sum_{k \in I \setminus I_\infty^{p()}} | a_k |^{p(k)} + \sup_{k \in I_\infty^{p()}} | a_k | \tag{5}$$

or

$$\rho_{l^{p()}(I)}(\{a_k\}_{k\in I}) = \left\| \left\{ \left| a^k \right|^{p(k)} \right\}_{k\in I\setminus I_\infty^{p()}} \right\|_{l^1(I\setminus I_\infty^{p()})} + \left\| \left\{ \left| a^k \right| \right\}_{k\in I_\infty^{p()}} \right\|_{l^\infty(I_\infty^{p()})}.$$

2. If $\left\{ \left| a^k \right|^{p(k)} \right\}_{k\in I\setminus I_\infty^{p()}} \notin l^1(I \setminus I_\infty^{p()})$ or $\left\{ \left| a^k \right| \right\}_{k\in I_\infty^{p()}}$ is unbounded on $I_\infty^{p()}$, we define:

$$\rho_{l^{p()}(I)}(\{a_k\}_{k\in I}) = \infty.$$

3. If $I_\infty^{p()} = \varnothing$, in particular when $p_+ < \infty$, we let $\sup_{k\in I_\infty^{p()}} | a_k | = 0$, therefore

$$\rho_{l^{p()}(I)}(\{a_k\}_{k\in I}) = \sum_{k\in I} | a_k |^{p(k)}.$$

4. If $I \setminus I_\infty^{p()} = \varnothing$, then $I = I_\infty^{p()}$, and $\rho_{p()}(\{a_k\}_{k\in I}) = \sup_{k\in I} | a_k |$.

5. If $p_+ < \infty$, then $p() < \infty$ on I and $I_\infty^{p()} = \varnothing$, therefore

$$\rho_{p(), I}(\{a_k\}_{k\in I}) = \sum_{k\in I} | a_k |^{p(k)}.$$

6. If $p_+ = \infty$, then $p()$ is unbounded on I and $I_\infty^{p()} = \{k \in I : p(k) = \infty\} = I$, and

$$\rho_{p(), I}(\{a_k\}_{k\in I}) = \sup_{k\in I} | a_k |.$$

7. For any $p() \in \mathcal{P}(I)$, we define the variable sequence spaces $l^{p()}(I)$ by:

$$l^{p()}(I) = \left\{ \{a_k\}_{k\in I} \in \mathbb{R}^I \ : \| \{a_k\}_{k\in I} \|_{l^{p()}(I)} < \infty \right\}, \tag{6}$$

where

$$\| \{a_k\}_{k\in I} \|_{l^{p()}(I)} = \inf\left\{ \lambda > 0 : \rho_{l^{p()}(I)}\left(\frac{\{a_k\}_{k\in I}}{\lambda} \right) \le 1 \right\}. \tag{7}$$

We define on $l^{p()}(I)$ some operations as follows:

For any $\{a_k\}_{k\in I} \in l^{p()}(I)$, $\{b_k\}_{k\in I} \in l^{p()}(I)$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R} \setminus \{0\}$:

$$\{a_k\}_{k\in I} + \{b_k\}_{k\in I} = \{a_k + b_k\}_{k\in I},$$

$$\alpha \cdot \{a_k\}_{k\in I} = \{\alpha \cdot a_k\}_{k\in I},$$

$$\{a_k\}_{k\in I} \cdot \{b_k\}_{k\in I} = \{a_k \cdot b_k\}_{k\in I},$$

$$\frac{\{a_k\}_{k\in I}}{\beta} = \left\{\frac{a_k}{\beta}\right\}_{k\in I}.$$

We also define the absolute value of any element $\{a_k\}_{k\in I}$ of $l^{p()}(I)$ by:

$$\left|\{a_k\}_{k\in I}\right| = \{| a_k |\}_{k\in I}.$$

The s-power of $\{a_k\}_{k\in I}$ of $l^{p()}(I)$ (with $1 \leq s < \infty$) is defined by;

$$\left|\{a_k\}_{k\in I}\right|^s = \{| a_k |^s\}_{k\in I}.$$

Properties 3. [22]

1) For any $p(), p() \in \mathcal{P}(I)$, $\{a_k\}_{k\in I} \in \mathbb{R}^I$:

$$p() \leq \tilde{p}() \quad \text{on} \quad I \Rightarrow \| \{a_k\}_{k\in I} \|_{l^{\tilde{p}()}(I)} \leq \| \{a_k\}_{k\in I} \|_{l^{p()}(I)}. \tag{8}$$

2) Given a non-void countable set I and $p() \in \mathcal{P}(I)$ such that $I_\infty^{p()} = \varnothing$, then for all s such that $\frac{1}{p_-} \leq s < \infty$, we have

$$\| \{| a_k |^s\}_{k\in I} \|_{l^{p()}(I)} = \| \{a_k\}_{k\in I} \|^s_{l^{sp()}(I)}.$$

3) When $p() = p$, $1 \leq p \leq \infty$, the definition of $l^{p()}(I)$ given in the formula (6) is equivalent to the classical Lebesgue sequence spaces $l^p(I)$ defined by:

$$\| (a_k)_{k\in I} \|_{l^p(I)} = \begin{cases} \left[\sum_{k\in I} | a_k |^p \right]^{\frac{1}{p}} & \text{if} \quad p < \infty, \\ \sup_{k\in I} | a_k | & \text{if} \quad p = \infty. \end{cases} \tag{9}$$

4) Given a countable and non-void set I and $p() \in \mathcal{P}(I)$, for all $\{a_k\}_{k\in I} \in l^{p()}(I)$ and $\{b_k\}_{k\in I} \in l^{p'()}(I)$, then $\{a_k b_k\}_{k\in I} \in l^1(I)$, and

$$\| \{a_k b_k\}_{k\in I} \|_{l^1(I)} = \sum_{k\in I} a_k b_k \le K_{p()} \| \{a_k\}_{k\in I} \|_{l^{p()}(I)} \times \| \{b_k\}_{k\in I} \|_{l^{p'()}(I)}. \tag{10}$$

This inequality can be generalized in the following way:

5) Given a countable and non-void set I and $q(), r() \in \mathcal{P}(I)$, define $p()$ by

$$\frac{1}{p()} = \frac{1}{q()} + \frac{1}{r()} \quad \text{on} \quad I.$$

Then there exists a constant K such that for all

$$\{a_k\}_{k\in I} \in l^{q()}(I),\ \{b_k\}_{k\in I} \in l^{r()}(I) : \{a_k b_k\}_{k\in I} \in l^{p()}(I) \text{ and}$$

$$\| \{a_k b_k\}_{k\in I} \|_{l^{p()}(I)} \le K_{p()} \| \{a_k\}_{k\in I} \|_{l^{q()}(I)} \times \| \{b_k\}_{k\in I} \|_{l^{r()}(I)}, \tag{11}$$

in the particular case where $p() = 1$, we get (10):

$$\| \{a_k b_k\}_{k\in I} \|_{l^1(I)} \le K \| \{a_k\}_{k\in I} \|_{l^{q()}(I)} \times \| \{b_k\}_{k\in I} \|_{l^{q'()}(I)}.$$

Lemma 4. [47, 48] *Let* $q() \in \mathcal{P}(\Omega)$, *suppose that* $q_+ < \infty$. *For any sequence* $\{f_n\} \subset L^{q()}(\Omega)$ *and* $f \in L^{q()}(\Omega)$, $\| f - f_n \|_{q(), \Omega} \to 0$ *if and only if* $\rho_{L^{q()}(\Omega)}(f - f_n) \to 0$.

Remark that the discrete version of this lemma is also valid.

Lemma 5. *Let* $(x, k, r) \in \Omega \times \mathbb{Z}^d \times]0, +\infty[$, *we set* $t = \frac{r}{2}$:

(a) $x \in I_k^t \Rightarrow I_k^t \subset J_x^r$,

(b) $x \in I_k^r \Rightarrow J_x^r \subset \bigcup_{l \in L_k} I_l^r,$

where $L_k = \{l \in \mathbb{Z}^d : k_j - 1 \leq l_j \leq k_j + 1,\ j \in \{1, ..., d\}\}$ *and*

(c) Card $L_k \leq 3^d < \infty.$

Proof.

(a) Under the hypotheses of (a) let $y \in I_k^t$, then $k_j t \leq y_j < (k_j + 1)t,$ $j = 1, ..., d,$ let us prove that $y \in J_x^r$:

by hypothesis $x \in I_k^t$, then $k_j t \leq x_j < (k_j + 1)t,$ these two last double inequalities lead to $-t < y_j - x_j < t,$ therefore $x_j - t < y_j < x_j$ $+t,$ $j = 1, ..., d,$ since $t = \frac{r}{2}$, we get $x_j - \frac{r}{2} < y_j < x_j + \frac{r}{2}$, $j = 1, ..., d,$ which means that $y \in J_x^r$.

(b) By hypothesis $x \in I_k^r$, then $k_j r \leq x_j < (k_j + 1)r,$ $j = 1, ..., d,$ let $z \in J_x^r$, then $x_j - \frac{r}{2} \leq z_j < x_j + \frac{r}{2}$, now we should find $l_0 \in L_k$ such that $z \in I_{l_0}^r$. $z \in I_{l_0}^r \Leftrightarrow l_{0_j} r \leq z_j < (l_{0_j} + 1)r,$ $j = 1, ..., d,$ this implies that $l_{0_j} \leq \frac{z_j}{r} < l_{0_j} + 1,$ for any real number x, let $E(x)$ be the great integer less than or equal to x, from the last double inequality, we can take $l_{0_j} = E\left(\frac{z_j}{r}\right)$, $j = 1, ..., d,$ then we get $l_0 = (l_{0_j})_{1 \leq j \leq d} = \left(E\left(\frac{z_j}{r}\right)\right)_{1 \leq j \leq d}$ such that $z \in I_{l_0}^r$, it remains to prove that $l_0 \in L_k$, that is, $k_j - 1 \leq l_{0_j} \leq k_j + 1,$ $j = 1, ..., d.$

By recapitulating we have: $\begin{cases} k_j r \leq x_j < (k_j + 1)r \\ x_j - \frac{r}{2} \leq z_j < x_j + \frac{r}{2} \end{cases}$, then we get:

$\begin{cases} k_j \leq \frac{x_j}{r} < k_j + 1 \\ \frac{x_j}{r} - \frac{1}{2} \leq \frac{z_j}{r} < \frac{x_j}{r} + \frac{1}{2} \end{cases}$, this implies that $\frac{x_j}{r} - \frac{1}{2} \leq \frac{z_j}{r} < \frac{x_j}{r} + \frac{1}{2}$ which gives $k_j - \frac{1}{2} \leq \frac{x_j}{r} - \frac{1}{2} \leq \frac{z_j}{r} < \frac{x_j}{r} + \frac{1}{2} < k_j + 1 + \frac{1}{2}$, that is, $k_j - \frac{1}{2} \leq \frac{z_j}{r} < k_j + 1 + \frac{1}{2}$, now by applying the function E (as defined in this proof) on all the members of this last double inequality, we find that $E\left(k_j - \frac{1}{2}\right) \leq E\left(\frac{z_j}{r}\right) < E\left(k_j + 1 + \frac{1}{2}\right)$, since $k_j \in \mathbb{Z}$, this is equivalent to $k_j - 1 \leq l_{0_j} \leq k_j + 1$, which means that $l_0 \in L_k$.

(c) Recall that for any integers a, b such that $a < b$ the number of integers located in the interval $[a, b]$ is $b - a + 1$, therefore the number of l_j such that $k_j - 1 \leq l_j \leq k_j + 1$ is 3, thus Card $L_k \leq 3^d$.

The claim is proved.

Definition 6. [22] Let $\mathbb{Z}^d \subset \Omega$, for any $q() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$, let $f \in L^{q()}_{loc}(\Omega)$ and $0 < r < \infty$:

$$_r\| f \|_{q(), p(), \Omega} = \left\| \left\{ \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}, \tag{12}$$

that is,

$$_r\| f \|_{q(), p(), \Omega} = \begin{cases} \inf\left\{ \lambda > 0 : \rho_{l^{p()}(\mathbb{Z}^d)}\left(\frac{\left\{ \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d}}{\lambda} \right) \leq 1 \right\} & \text{if} \quad p_+ < \infty, \\ \sup_{k \in \mathbb{Z}^d} \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} & \text{if} \quad p_+ = \infty. \end{cases} \tag{13}$$

We have defined the two-variable exponential amalgam spaces

$(L^{q()}, l^{p()})(\Omega)$ (see [22]) by:

$$(L^{q()}, l^{p()})(\Omega) = \left\{ f \in L_{loc}^{q()} \ : \ {}_1\| f \|_{q(), p(), \Omega} < \infty \right\}. \tag{14}$$

${}_r\| f \|_{q(), p(), \mathbb{R}^d}$, $(L^{q()}, l^{p()})(\mathbb{R}^d)$ will be simply denoted by ${}_r\| f \|_{q(), p()}$, $(L^{q()}, l^{p()})$.

Remark that (13) generalizes (1) and according to Lemma 5, when $p_+ = \infty$, I_k^r can be replaced by J_x^r and so (13) becomes:

$${}_r\| f \|_{q(), p(), \Omega} = \begin{cases} \inf\left\{ \lambda > 0 : \rho_{l^{p()}(\mathbb{Z}^d)}\left(\dfrac{\left\{ \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d}}{\lambda} \right) \leq 1 \right\} & \text{if} \quad p_+ < \infty, \\ \sup\limits_{x \in \Omega} \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega)} & \text{if} \quad p_+ = \infty. \end{cases} \tag{15}$$

Proposition 7. [22]

1. *Let* $\mathbb{Z}^d \subset \Omega$, *given* $q() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$. $(L^{q()}, l^{p()})(\Omega)$, ${}_1\| \cdot \|_{q(), p()}$ *is a Banach space.*

2. *Let* Ω *be a set such that* $\mathbb{Z}^d \subset \Omega$, *given* $q(), q_1(), q_2() \in \mathcal{P}(\Omega)$, $p(), p_1(), p_2() \in \mathcal{P}(\mathbb{Z}^d)$ *and* $f \in L_{loc}^{q()}(\Omega)$.

a) *If* $\max\left\{ \dfrac{1}{q_-}, \dfrac{1}{p_-} \right\} \leq s < \infty$, $\left| \Omega_\infty^{q()} \right| = 0$, *and* $f \in (L^{q()}, l^{p()})(\Omega)$, *then* ${}_1\left\| | f |^s \right\|_{q(), p() \Omega} = {}_1\| f \|_{sq(), sp(), \Omega}^s$.

b) $(\cdot)$ *Let* $f \in (L^{q_2()}, l^{p()})(\Omega)$. *If* $q_1() \leq q_2()$ *on* Ω, *then* $f \in (L^{q_1()}, l^{p()})(\Omega)$ *and* ${}_1\| f \|_{q_1(), p(), \Omega} \leq K_{q_1(), q_2()} \times {}_1\| f \|_{q_2(), p(), \Omega}$ *or* $(L^{q_2()}, l^{p()})(\Omega) \subset (L^{q_1()}, l^{p()})(\Omega)$.

(••) *In particular when* $\left|\Omega \setminus \Omega_{\infty}^{q_1()}\right| < \infty$, *we have*

$$_1\| f \|_{q_1(), p(), \Omega} \leq \left(1 + \left| \Omega \setminus \Omega_{\infty}^{q_1()} \right|\right) \cdot {}_1\| f \|_{q_2(), p(), \Omega} \leq (1 + |\Omega|) \cdot {}_1\| f \|_{q_2(), p(), \Omega}.$$

c) • *Let* $f \in (L^{q()}, l^{p_1()})(\Omega)$ *and* $p_1() \leq p_2()$ *on* $\mathbb{Z}^d$. *We have* $f \in (L^{q()}, l^{p_2()})(\Omega)$ *and* $_1\| f \|_{q(), p_2(), \Omega} \leq C \times {}_1\| f \|_{q(), p_1(), \Omega}$ *or* $(L^{q()}, l^{p_1()})(\Omega) \subset (L^{q()}, l^{p_2()})(\Omega)$.

•• *In particular when* $|\Omega| < \infty$, *we have*

$$_1\| f \|_{q(), p(), \Omega} \leq (1 + |\Omega|) \cdot {}_1\| f \|_{q_+, p_-, \Omega}.$$

d) *If* $\begin{cases} q() = q = \text{constant real}, \\ p() = p = \text{constant real}, \end{cases}$

both in $[1, \infty]$, *then* $(L^{q()}, l^{p()})(\Omega) = (L^q, l^p)(\Omega)$ *with constant exponents.*

$(L^q, l^p)(\Omega)$ *with constant exponents have been widely studied by many researchers* (*see*: [45], [24], [28], [25], [46].

e) *If* $|\Omega| < \infty$, *then there exist positive constant reals* c, C *such that*:

$$c \times {}_1\| f \|_{q_-, p_+, \Omega} \leq {}_1\| f \|_{q(), p(), \Omega} \leq C \times {}_1\| f \|_{q_+, p_-, \Omega}$$

otherwise

$$(L^{q_+}, l^{p_-})(\Omega) \subset (L^{q()}, l^{p()})(\Omega) \subset (L^{q_-}, l^{p_+})(\Omega).$$

f) *If*

$$\frac{1}{q_1()} + \frac{1}{q_2()} = \frac{1}{q()} \leq 1,$$

$$\frac{1}{p_1()} + \frac{1}{p_2()} = \frac{1}{p()} \leq 1,$$

$$(f, g) \in (L^{q_1()}, l^{p_1()})(\Omega) \times (L^{q_2()}, l^{p_2()})(\Omega).$$

Then

$$\begin{cases} fg \in (L^{q()}, l^{p()})(\Omega), \\ {}_1\| fg \|_{q(), p(), \Omega} \leq C \times {}_1\| f \|_{q_1(), p_1(), \Omega} \times {}_1\| f \|_{q_2(), p_2(), \Omega}, \end{cases}$$

in the particular case where $q() = p() = 1$ *on* Ω, *we have Hölder's inequality*

$${}_1\| fg \|_{1, 1\Omega} \leq C \times {}_1\| f \|_{q(), p(), \Omega} \times {}_1\| g \|_{q'(), p'(), \Omega}.$$

g) *Let* $f \in (L^{q()}, l^{p()})(\Omega)$. *We have*

$${}_1\| f \|_{q_+ + q_-, p(), \Omega} \leq C(p())[{}_1\| f_1 \|_{q_+, p(), \Omega} + {}_1\| f_2 \|_{q_-, p(), \Omega}] \leq 2 \times {}_1\| f \|_{q(), p(), \Omega}$$

with $f_1 \in (L^{q_+}, l^{p()})(\Omega)$ *and* $f_2 \in (L^{q_-}, l^{p()})(\Omega)$, *otherwise*

$$(L^{q()}, l^{p()})(\Omega) \subset (L^{q_+}, l^{p()})(\Omega) + (L^{q_-}, l^{p()})(\Omega) \subset (L^{q_+ + q_-}, l^{p()})(\Omega).$$

h) *For any* $r > 0$, *the norms* ${}_1\| \cdot \|_{q(), p()}$ *and* ${}_r\| \cdot \|_{q(), p()}$ *are equivalent.*

We will need the following theorem which can be found in [47].

Theorem 8. (**Monotone Convergence theorem**)

Let $q() \in \mathcal{P}(\Omega)$, *and* $(f_n)_{n \in \mathbb{N}} \subset L^{q()}(\Omega)$ *be a sequence of non-negative functions such that* f_n *increases to a function* f *pointwise almost everywhere. Then either* $f \in L^{q()}(\Omega)$ *and* $\| f_n \|_{L^{q()}(\Omega)} \to \| f \|_{L^{q()}(\Omega)}$ *or* $f \notin L^{q()}(\Omega)$ *and* $\| f_n \|_{L^{q()}(\Omega)} \to \infty$.

Remark 9.

(1) Let $p() \in LH(\Omega)$, there exists a function $\widetilde{p}() \in \mathcal{P}(\mathbb{R}^d)$ such that

(a) $\widetilde{p}() \in LH$;

(b) $\widetilde{p}(x) = p(x)$, $x \in \Omega$;

(c) $\widetilde{p}_- = p_-$ and $\widetilde{p}_+ = p_+$.

(2) Given two domains Ω and $\widetilde{\Omega}$, if $\widetilde{\Omega} \subset \Omega$ and $p() \in \mathcal{P}(\Omega)$, then

$\widetilde{p}() = p()|_{\widetilde{\Omega}} \in \mathcal{P}(\widetilde{\Omega})$ and it is immediate from the definition of the norm $\|\cdot\|_{L^{q()}(\Omega)}$ that for $f \in L^{p()}(\Omega)$,

$$\| f \|_{L^{\widetilde{p}()}(\widetilde{\Omega})} = \| f\chi_{\widetilde{\Omega}} \|_{L^{p()}(\Omega)}.$$

Hereafter we will implicitly make these restrictions without comment and simply write $\| f \|_{L^{p()}(\widetilde{\Omega})}$. Conversely, given $p() \in \mathcal{P}(\widetilde{\Omega})$ and $f \in L^{p()}(\widetilde{\Omega})$, we can extend both to Ω by defining $f(x) = 0$ for $x \in \Omega \setminus \widetilde{\Omega}$ and defining $p()$ arbitrarily on $\Omega \setminus \widetilde{\Omega}$. If we do so, then $\| f \|_{L^{p()}(\widetilde{\Omega})} = \| f \|_{L^{p()}(\Omega)}$. Moreover, if $p() \in LH(\widetilde{\Omega})$, by (1) we may assume that $p() \in LH(\Omega)$ as well.

Combining (1) and (2); we deduct:

(3) Let Ω be a set such that $\mathbb{Z}^d \subset \Omega$, from (1) and (2) given $t() \in LH(\mathbb{Z}^d)$, we can extend $t()$ to an exponent function $\widetilde{t}()$ in $LH(\Omega)$: in the following sense:

$$\widetilde{t}(x) = \begin{cases} t(x) & \text{if} \quad x \in \mathbb{Z}^d, \\ 0 & \text{if} \quad x \in \Omega \setminus \mathbb{Z}^d. \end{cases}$$

We will not do a difference between $\widetilde{t}()$ and $t()$.

We will need the following lemma:

Lemma 10. [22]

(i) *For a fixed positive real number* r, *if* $k_1, k_2 \in \mathbb{Z}^d$ *and* $k_1 \neq k_2$, *then* $I_{k_1}^r \cap I_{k_2}^r = \varnothing$.

(ii) *If* r *is a fixed positive real number then* $\bigcup_{k \in \mathbb{Z}^d} I_k^r = \Omega$.

Lemma 11. *Let* A *be a finite subset of* $\mathbb{Z}^d$ *such that* $\mathbb{Z}^d \subset \Omega$.

a) *There always exist nonnegative real numbers* $(m_k)_{k \in A}$ *not all zero* *such that*

$$\sum_{k \in A} m_k \chi_{I_k^r}(x) = 0, \quad x \in \Omega.$$

b) *One element of* $\left(\chi_{I_k^r}\right)_{k \in A}$ *can be expressed from another.*

c) *Furthermore all elements of* $\left(\chi_{I_k^r}\right)_{k \in A}$ *can be expressed from one of them.*

Proof.

a) Recall that for real numbers $(a_k)_{k \in A} : \left| \sum a_k \right| = \sum | a_k |$ when the numbers have the same sign.

For vectors $(a_k)_{k \in A}$ (with A the finite subset of $\mathbb{Z}^d$) in normed vector space $(\nu, \| \cdot \|) : \left\| \sum_k a_k \right\| = \sum_k \| a_k \|$ when the vectors a_k are parallel or collinear with the same direction otherwise there exit real numbers r_k not all zero such that $\sum_{k \in A} r_k a_k = 0$, if the vectors a_k are collinear with the same sense, all the vectors a_k can be expressed from one of them, which means that there exist $k_0 \in A$ and nonnegative real numbers $(m_k)_{k \in A - \{k_0\}}$ such that $\forall k \in A - \{k_0\} : a_k = m_k a_{k_0}$, in this case:

$$\begin{aligned}
\left\| \sum_{k \in A} a_k \right\| &= \left\| a_{k_0} + \sum_{k \in A - \{k_0\}} a_k \right\| \\
&= \left\| a_{k_0} + \sum_{k \in A - \{k_0\}} m_k a_{k_0} \right\| \\
&= \left\| \left(1 + \sum_{k \in A - \{k_0\}} m_k \right) a_{k_0} \right\|
\end{aligned}$$

$$= \left(1 + \sum_{k \in A - \{k_0\}} m_k\right) \| a_{k_0} \|$$

in other hand

$$\begin{aligned} \sum_{k \in A} \| a_k \| &= \| a_{k_0} \| + \sum_{k \in A - \{k_0\}} \| a_k \| \\ &= \| a_{k_0} \| + \sum_{k \in A - \{k_0\}} \| m_k a_{k_0} \| \\ &= \| a_{k_0} \| + \sum_{k \in A - \{k_0\}} | m_k | \| a_{k_0} \| \\ &= \left(1 + \sum_{k \in A - \{k_0\}} | m_k |\right) \| a_{k_0} \| \\ &= \left(1 + \sum_{k \in A - \{k_0\}} m_k\right) \| a_{k_0} \|. \end{aligned}$$

The last equality is from the fact that $(m_k)_{k \in A - \{k_0\}}$ are nonnegative real numbers. Therefore, if the vectors $(a_k)_{k \in A}$ are collinear with the same direction then

$$\left\| \sum_{k \in A} a_k \right\| = \sum_{k \in A} \| a_k \|.$$

From Lemma 10, for any $k_1, k_2 \in A : k_1 \neq k_2 \Rightarrow I_{k_1}^r \cap I_{k_2}^r = \varnothing$, therefore the members of the family $\left(\chi_{I_k^r}\right)_{k \in A}$ are pairwise disjoint.

The members of the family $\left(\chi_{I_k^r}\right)_{k \in A}$ are collinear with the same sense, let us prove that: let $x \in \Omega$ such that

$$\sum_{k \in A} m_k \chi_{I_k^r}(x) = 0.$$

Taking into account of the disjointness of the family, we have two cases:

First case: $x \in \Omega \setminus \bigcup_{k \in A} I_k^r$.

In this case, we can take $m_k = 1$ for any $k \in A$ and (16) holds.

Second case: There exists a unique $k_0 \in A$ such that $x \in I_{k_0}^r$.

In this case (16) will be reduced to $m_{k_0} \chi_{I_{k_0}^r}(x) = 0$, we can take $m_k = 0$ and for any $k \in A \setminus \{k_0\} : m_k = 1$ and (16) holds, otherwise $\left(\chi_{I_k^r}\right)_{k \in A}$ are collinear and we can remark that $(m_k)_{k \in A}$ are also nonnegative, therefore $\left(\chi_{I_k^r}\right)_{k \in A}$ are collinear with the same sense.

b) Let us prove that one element of $\left(\chi_{I_k^r}\right)_{k \in A}$ can be expressed from another.

We determine two nonnegative real numbers m_{k_i} and m_{k_j}

$$m_{k_i} \chi_{I_{k_i}^r}(x) = m_{k_j} \chi_{I_{k_j}^r}(x), \quad x \in \Omega. \tag{e}$$

If $x \notin I_{k_i}^r \cup I_{k_j}^r$, then we can take $m_{k_i} = m_{k_j} = 1$ and then (e) becomes $\chi_{I_{k_i}^r}(x) = \chi_{I_{k_j}^r}(x) = 0$. If x is in one of the two sets $I_{k_i}^r$ and $I_{k_j}^r$, for instance $x \in I_{k_i}^r \setminus I_{k_j}^r$, then we can take $m_{k_i} = 0$ and $m_{k_j} = 1$, therefore (e) becomes $\chi_{I_{k_j}^r}(x) = \frac{m_{k_i}}{m_{k_j}} \chi_{I_{k_i}^r}(x) = a \cdot \chi_{I_{k_i}^r}(x)$ with $a = \frac{m_{k_i}}{m_{k_j}}$. In any case $\chi_{I_{k_j}^r} = a \cdot \chi_{I_{k_i}^r}$ with $a \geq 0$.

c) To be simple let $A = \{k_1, \ldots, k_n\}$ with $n \geq 1$ and

$S = \left\{ \chi_{I_{k_1}^r}, \chi_{I_{k_2}^r}, \ldots, \chi_{I_{k_n}^r} \right\}$, if we apply b): there will exist real numbers $a_2, a_3, \ldots, a_n$ such that $\chi_{I_{k_1}^r} = 1 \cdot \chi_{I_{k_1}^r}$, $\chi_{I_{k_2}^r} = a_2 \cdot \chi_{I_{k_1}^r}$, $\chi_{I_{k_3}^r} = a_3 \cdot \chi_{I_{k_1}^r}$, ..., $\chi_{I_{k_n}^r} = a_n \cdot \chi_{I_{k_1}^r}$, then S will be:

$$S = \left\{ \chi_{I_{k_1}^r}, a_2\chi_{I_{k_1}^r}, a_3\chi_{I_{k_1}^r}, \ldots, a_n\chi_{I_{k_1}^r} \right\}.$$

Finally we see that S is generated only by one vector $\chi_{I_{k_1}^r}$.

Consequence of Lemma 11 (Cons. Lem. 11)

The consequence of this lemma is that for any $p() = \mathcal{P}(\Omega)$ with $\mathbb{Z}^d \in \Omega$ and A a finite subset of $\mathbb{Z}^d$, for all $f \in L^0(\Omega)$, we have:

$$\left\| \sum_{k \in A} \chi_{I_k^r} \right\| = \sum_{k \in A} \left\| \chi_{I_k^r} \right\| \tag{f1}$$

but in other hand we have

$$\left\| \sum_{k \in A} \chi_{I_k^r} \right\| = \left\| \chi \bigcup_{k \in A} I_k^r \right\|, \tag{f2}$$

therefore

$$\left\| \chi \bigcup_{k \in A} I_k^r \right\| = \sum_{k \in A} \left\| \chi_{I_k^r} \right\|, \tag{f3}$$

$$\| f \|_{L^{p()}(\Omega)} \overset{\text{lem.10-(ii)}}{=} \| f \|_{L^{p()}\left(\bigcup_{k \in \mathbb{Z}^d} I_k^r \right)} = \left\| f\chi_{\bigcup_{k \in \mathbb{Z}^d} I_k^r} \right\|_{L^{p()}(\Omega)}$$

$$= \left\| \sum_{k \in \mathbb{Z}^d} f\chi_{I_k^r} \right\|_{L^{p()}(\Omega)} = \sum_{k \in \mathbb{Z}^d} \left\| f\chi_{I_k^r} \right\|_{L^{p()}(\Omega)},$$

that is,

$$\sum_{k\in\mathbb{Z}^d}\left\| f\chi_{I_k^r}\right\|_{L^{p()}(\Omega)} = \left\| f\chi_{\bigcup_{k\in\mathbb{Z}^d} I_k^r}\right\|_{L^{p()}(\Omega)} = \| f\|_{L^{p()}(\Omega)},$$

$$I_k^r \subset \Omega, \quad k \in \mathbb{Z}^d, \quad r > 0. \tag{f}$$

The following propositions on $(L^{q()}, l^{p()})(\Omega)$ are not proved in [22], we will do it in this work:

Proposition 12. *Let* $\mathbb{Z}^d \subset \Omega$, *given* $q() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$ *and* $f \in L^{q()}_{loc}(\Omega)$.

If we simply let $s() = q() = p()$ on Ω, *then*

$$(L^{s()}, l^{s()})(\Omega) = L^{s()}(\Omega),$$

that is, there exist two constants K *and* C *such that*

$$K\| f\|_{L^{s()}(\Omega)} \leq {}_r\| f\|_{s(), s(), \Omega} \leq C \cdot \| f\|_{L^{s()}(\Omega)}.$$

Proof.

First case: $s_+ < \infty$

In this case, for any x of $\Omega : s(x) < \infty$ and for any $\{a_k\}_{k\in\mathbb{Z}^d}$ of $\mathbb{R}^{\mathbb{Z}^d}$, we also have: $\rho_{l^{s()}(\mathbb{Z}^d)}(\{a_k\}_{k\in\mathbb{Z}^d}) = \sum_{k\in\mathbb{Z}^d}|a_k|^{s(k)}$, therefore

$${}_r\| f\|_{s(), s(), \Omega} = \left\|\left\{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{s()}(\mathbb{Z}^d)}$$

$$= \inf\left\{\lambda > 0 : \rho_{l^{s()}(\mathbb{Z}^d)}\left(\frac{\left\{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}}{\lambda}\right) \leq 1\right\}$$

$$= \inf\left\{\lambda > 0 : \sum_{k\in\mathbb{Z}^d}\left(\frac{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}}{\lambda}\right)^{s(k)} \leq 1\right\}.$$

For any $(k, r, \lambda) \in \mathbb{Z}^d \times (0, \infty) \times (0, \infty)$, we have:

$$\left(\frac{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}}{\lambda}\right)^{s(k)} \leq \sum_{k\in\mathbb{Z}^d}\left(\frac{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}}{\lambda}\right)^{s(k)},$$

from this we get

$$\left\{\lambda > 0 : \sum_{k\in\mathbb{Z}^d}\left(\frac{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}}{\lambda}\right)^{s(k)} \leq 1\right\} \subset \left\{\lambda > 0 : \left(\frac{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}}{\lambda}\right)^{s(k)} \leq 1\right\},$$

then we have

$$\inf\left\{\lambda > 0 : \left(\frac{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}}{\lambda}\right)^{s(k)} \leq 1\right\} \leq \inf\left\{\lambda > 0 : \sum_{k\in\mathbb{Z}^d}\left(\frac{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}}{\lambda}\right)^{s(k)} \leq 1\right\}.$$

Now we consider the left hand side term $\inf\left\{\lambda > 0 : \left(\frac{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}}{\lambda}\right)^{s(k)} \leq 1\right\}$:

$$\inf\left\{\lambda > 0 : \left(\frac{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}}{\lambda}\right)^{s(k)} \leq 1\right\} = \inf\left\{\lambda > 0 : \frac{\left\| f\chi_{I_k^r}\right\|_{L^{s()}(\Omega)}}{\lambda} \leq 1\right\}$$

$$= \left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)}.$$

The right hand side term $\inf\left\{\lambda > 0 : \sum_{k\in\mathbb{Z}^d}\left(\frac{\left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)}}{\lambda}\right)^{s(k)} \leq 1\right\}$ is $_r\| f \|_{s(), s(), \Omega}$ (see the line 4th and 5th of the current proof), therefore

$$\left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)} \leq {}_r\| f \|_{s(), s(), \Omega}. \tag{17}$$

First we prove that $(L^{s()}, l^{s()})(\Omega) \subset L^{s()}(\Omega)$:

Let $f \in (L^{s()}, l^{s()})(\Omega)$, therefore $M = {}_r\| f \|_{s(), s(), \Omega} < \infty$, this last equality combined with (17) leads to

$$\left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)} \leq M < \infty. \tag{18}$$

Remark that for any $r > 0$, we have:

$[-r, r)^d = \bigcup_{k\in\{-1, 0\}^d} I_k^r$ and $\{-1, 0\}^d$ owns 2^d members, thus

$$\left\| f\chi_{[-r, r[^d} \right\|_{L^{s()}(\Omega)} = \left\| \sum_{k\in\{-1, 0\}^d} f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)} \leq \sum_{k\in\{-1, 0\}^d} \left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)}$$

$$\overset{(18)}{\leq} \sum_{k\in\{-1, 0\}^d} M = 2^d M,$$

therefore for any $n \in \mathbb{N}$: $\left\| f\chi_{[-n, n)^d} \right\|_{L^{s()}(\Omega)} \leq 2^d M$ or $\left(| f |^{s()} \chi_{[-n, n)^d}\right) \uparrow | f |^{s()}$, from the Monotone Convergence Theorem 8 (in variable Lebesgue spaces) we have:

$$\| f \|_{L^{s()}(\Omega)} = \lim_{n\to\infty} \left\| f\chi_{[-n,n)^d} \right\|_{L^{s()}(\Omega)} \le 2^d M = 2^d \cdot {}_r\| f \|_{s(), s(), \Omega},$$ that is,

$$\| f \|_{L^{s()}(\Omega)} \le 2^d \cdot {}_r\| f \|_{s(), s(), \Omega}. \qquad (i_1)$$

Secondly, we prove that $L^{s()}(\Omega) \subset (L^{s()}, l^{s()})(\Omega)$:

since

$$1 \le s(k) < \infty \text{ and } \sum_{k\in\mathbb{Z}^d} \left(\frac{\left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)}}{\lambda} \right)^{s(k)} \le \left(\sum_{k\in\mathbb{Z}^d} \frac{\left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)}}{\lambda} \right)^{s(k)}, \quad (i_2)$$

$$\begin{aligned}
{}_r\| f \|_{s(), s(), \Omega} &= \left\| \left\{ \left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{s()}(\mathbb{Z}^d)} \\
&= \inf\left\{ \lambda > 0 : \rho_{l^{s()}(\mathbb{Z}^d)} \left(\frac{\left\{ \left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)} \right\}_{k\in\mathbb{Z}^d}}{\lambda} \right) \le 1 \right\} \\
&= \inf\left\{ \lambda > 0 : \sum_{k\in\mathbb{Z}^d} \left(\frac{\left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)}}{\lambda} \right)^{s(k)} \le 1 \right\} \\
&\le \inf\left\{ \lambda > 0 : \left(\sum_{k\in\mathbb{Z}^d} \frac{\left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)}}{\lambda} \right)^{s(k)} \le 1 \right\} \quad \text{from } (i_2) \\
&= \inf\left\{ \lambda > 0 : \sum_{k\in\mathbb{Z}^d} \frac{\left\| f\chi_{I_k^r} \right\|_{L^{s()}(\Omega)}}{\lambda} \le 1 \right\}
\end{aligned}$$

$$= \inf\left\{\lambda > 0 : \frac{\left\| f\chi_{\bigcup_{k\in\mathbb{Z}^d} I_k^r} \right\|_{L^{s()}(\Omega)}}{\lambda} \leq 1\right\} \quad \text{Cons. of lem. 11}$$

$$= \inf\left\{\lambda > 0 : \frac{\| f\chi_\Omega \|_{L^{s()}(\Omega)}}{\lambda} \leq 1\right\} \quad \text{by lem. 10}$$

$$= \inf\left\{\lambda > 0 : \frac{\| f \|_{L^{s()}(\Omega)}}{\lambda} \leq 1\right\}$$

$$= \| f \|_{L^{s()}(\Omega)},$$

that is,

$$_r\| f \|_{s(), s(), \Omega} \leq \| f \|_{L^{s()}(\Omega)}. \tag{i_3}$$

(i_1) and (i_3) imply

$$\| f \|_{L^{q()}(\Omega)} \leq 2^d \cdot {}_r\| f \|_{s(), s(), \Omega} \leq 2^d \| f \|_{L^{s()}(\Omega)}.$$

Second case: $s_+ = \infty$

In this case $s()$ is unbounded, then we can tend $s()$ to ∞, that is, $\lim_{s()\to\infty} {}_r\| f \|_{s(), s(), \Omega} = {}_r\| f \|_{\infty, \infty, \Omega}$ and $I_\infty^{s()} = \{k \in I : s(k) = \infty\} = I$, and $\rho_{s()}(\{a_k\}_{k\in I}) = \sup_{k\in I} | a_k |$, then we have:

$$_r\| f \|_{\infty, \infty, \Omega} = \inf\left\{\lambda > 0 : \rho_{l^\infty(\mathbb{Z}^d)}\left(\frac{\left\{\left\| f\chi_{I_k^r} \right\|_{L^\infty(\Omega)}\right\}_{k\in\mathbb{Z}^d}}{\lambda}\right) \leq 1\right\}$$

$$= \inf\left\{\lambda > 0 : \sup_{k\in\mathbb{Z}^d} \frac{\left\| f\chi_{I_k^r} \right\|_{L^\infty(\Omega)}}{\lambda} \leq 1\right\}$$

$$= \inf\left\{\lambda > 0 : \frac{\| f \|_{L^\infty(\Omega)}}{\lambda} \leq 1\right\}$$

$$= \inf\{\lambda > 0 : \lambda \geq \| f \|_{L^\infty(\Omega)}\}$$

$$= \| f \|_{L^\infty(\Omega)}.$$

Therefore $(L^{s()}, l^{s()})(\Omega) = L^{s()}(\Omega)$.

Proposition 13. *Let* $\mathbb{Z}^d \subset \Omega$, *given* $q() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$ *and* $f \in L_{loc}^{q()}(\Omega)$.

(a) $q() \leq p() \Rightarrow L^{q()}(\Omega) \cup L^{p()}(\Omega) \subset (L^{q()}, l^{p()})(\Omega)$, *that is, there exists a positive constant* C *such that*

$$_r\| f \|_{q(), p(), \Omega} \leq C \cdot \min\{\| f \|_{L^{q()}(\Omega)}, \| f \|_{L^{p()}(\Omega)}\}. \tag{19}$$

(b) $p() \leq q() \Rightarrow (L^{q()}, l^{p()})(\Omega) \subset L^{q()}(\Omega) \cap L^{p()}(\Omega)$, *that is, there exists a positive constant* C *such that*

$$\max\{\| f \|_{L^{q()}(\Omega)}, \| f \|_{L^{p()}(\Omega)}\} \leq C \cdot {}_r\| f \|_{q(), p(), \Omega}. \tag{20}$$

Proof.

(a) First

Recall that in Proposition 7-h) it is said that:

For any $r > 0$, the norms $_1\| \cdot \|_{q(), p()}$ and $_r\| \cdot \|_{q(), p()}$ are equivalent. $q() \leq p() \Rightarrow \frac{1}{q()} \geq \frac{1}{p()}$, this implies that there exists a function $\alpha()$ defined on Ω such that $\frac{1}{q()} = \frac{1}{p()} + \frac{1}{\alpha()}$, by Hölder's inequality we have:

$$\left\| f\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} = \left\| \left(f\chi_{I_k^1} \right) \chi_{I_k^1} \right\|_{L^{q()}(\Omega)} \leq C \left\| f\chi_{I_k^1} \right\|_{L^{p()}(\Omega)} \left\| \chi_{I_k^1} \right\|_{L^{\alpha()}(\Omega)}$$

$$\overset{\text{Lem. 2.39 of [47]}}{\Rightarrow} \left\| f\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} \leq 2C \left\| f\chi_{I_k^1} \right\|_{L^{p()}(\Omega)}$$

$$\Rightarrow \left\| \left\{ \left\| f\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \leq 2C \left\| \left\{ \left\| f\chi_{I_k^1} \right\|_{L^{p()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)},$$

that is,

$${}_1\| f \|_{q(), p(), \Omega} \leq 2C \left\| \left\{ \left\| f\chi_{I_k^1} \right\|_{L^{p()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$= K \cdot {}_1\| f \|_{p(), p()\, \Omega} \overset{\text{Propo. 12}}{=} K \| f \|_{L^{p()}(\Omega)},$$

that is,

$$L^{p()}(\Omega) \subset (L^{q()}, l^{p()})(\Omega).$$

Secondly

$$\left\| \left\{ \left\| f\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} = {}_1\| f \|_{q(), p()} \overset{\text{Propo 7-2)-c)}}{\leq} {}_1\| f \|_{q(), q()} \overset{\text{Propo.12}}{=} \| f \|_{L^{q()}(\Omega)},$$

that is,

$$L^{q()}(\Omega) \subset (L^{q()}, l^{p()})(\Omega).$$

These two results yield to

$$L^{q()}(\Omega) \cup L^{p()}(\Omega) \subset (L^{q()}, l^{p()})(\Omega).$$

(b) $p() \leq q()$

First

$\| f \|_{L^{q()}(\Omega)} \overset{\text{Propo. 12}}{=} {}_r\| f \|_{q(), q()\, \Omega} \overset{\text{Propo. 7-2)-c)}}{\leq} {}_r\| f \|_{q(), p()\, \Omega}$ which means

$(L^{q()}, l^{p()})(\Omega) \subset L^{q()}(\Omega).$

Secondly

$\| f \|_{L^{p()}(\Omega)} = {}_r\| f \|_{p(), p() \Omega} \overset{\text{Propo. 7-2)-b)}}{\leq} {}_r\| f \|_{q(), p() \Omega}$ which means

$(L^{q()}, l^{p()})(\Omega) \subset L^{p()}(\Omega).$

These two results yield to

$$(L^{q()}, l^{p()})(\Omega) \subset L^{q()}(\Omega) \cap L^{p()}(\Omega).$$

Definition 14. Let $\mathbb{Z}^d \subset \Omega$, for any $q(), \alpha() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$.

For any $f \in L^{q()}_{loc}(\Omega)$ and $0 < r < \infty$, we define $\| f \|_{q(), p(), \alpha(), \Omega}$ by:

$$\| f \|_{q(), p(), \alpha(), \Omega} = \sup_{\substack{r>0 \\ x\in\Omega}} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|_{q(), p(), \Omega}. \tag{21}$$

We define the spaces $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$ by:

$$(L^{q()}, l^{p()})^{\alpha()}(\Omega) = \left\{ f \in L^{q()}_{loc}(\Omega) : \| f \|_{q(), p(), \alpha(), \Omega} < \infty \right\}. \tag{22}$$

Remark 15.

1. From (21) we have $r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|_{q(), p(), \Omega} \leq \| f \|_{q(), p(), \alpha(), \Omega}$, $x \in \Omega$, $r > 0$, in particular when $r = 1$, we have ${}_1\| f \|_{q(), p(), \Omega} \leq \| f \|_{q(), p(), \alpha(), \Omega}$ which means that

$$(L^{q()}, l^{p()})^{\alpha()}(\Omega) \subset (L^{q()}, l^{p()})(\Omega).$$

2. In case of $p_+ = \infty$, by virtue of (15), (21) becomes $\| f \|_{q(), p(), \alpha(), \Omega} = \sup_{\substack{r>0 \\ x\in\Omega}} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \| f\chi_{J_x^r} \|_{L^{q()}(\Omega)}$, in this case the

exponent function $p()$ is unbounded and we can tend $p()$ to ∞, the last equality becomes

$$\| f \|_{q(),\infty,\alpha(),\Omega} = \sup_{\substack{r>0\\ x\in\Omega}} r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)} \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega)}. \tag{23}$$

From this last inequality we will get

$$\left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega)} \leq r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),\infty,\alpha(),\Omega}, \quad r > 0, \quad x \in \Omega. \tag{24}$$

Again, in case of $p_+ = \infty$, by virtue of (13), (21) becomes

$$\| f \|_{q(),\infty,\alpha(),\Omega} = \sup_{\substack{k\in\mathbb{Z}^d\\ r>0\\ x\in\Omega}} r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)} \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)}. \tag{25}$$

From this we get

$$\left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \leq r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),\infty,\alpha(),\Omega}, \ k \in \mathbb{Z}^d, \ r > 0, \ x \in \Omega. \tag{26}$$

3. $(L^{q()}, l^{\infty})^{\alpha()}(\Omega)$ generalize the classical Morrey space

$$L^{q,\lambda}(\Omega) = \{f \in L^0(\Omega) : \| f \|_{q,\lambda,\Omega} < \infty\},$$

where

$\| f \|_{q,\lambda,\Omega} = \sup_{r>0,\, x\in\Omega} r^{-d\lambda} \left\| f\chi_{J_x^r} \right\|_{L^q(\Omega)}$ and $\lambda = \frac{1}{\alpha} - \frac{1}{q}$, λ, q, p are all constants.

Remark 16. By definition of $\| \cdot \|_{q(),p(),\alpha(),\Omega}$ we have:

$$_r\| f \|_{q(),p(),\Omega} \leq r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),p(),\alpha(),\Omega}, \quad r > 0, \quad x \in \Omega. \tag{27}$$

But in other hand $(L^{q()}, l^{p()})(\Omega) \subset L^{q()}_{loc}(\Omega)$.

By Proposition 7-2-(c), we have ${}_r\| f \|_{q(),\infty,\Omega} \leq {}_r\| f \|_{q(),p(),\Omega}$, that is,

$$\left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \leq {}_r\| f \|_{q(),p(),\Omega}, \quad k \in \mathbb{Z}^d, \quad r > 0 \tag{28}$$

or

$$\left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega)} \leq {}_r\| f \|_{q(),p(),\Omega}, \quad x \in \Omega, \quad r > 0 \tag{29}$$

or more generally for any cube Q such that $|Q| < \infty$ we have

$$\| f\chi_Q \|_{L^{q()}(\Omega)} \leq {}_r\| f \|_{q(),p(),\Omega}, \quad \text{with } r = |Q|^{1/d}.$$

Combining (27), (28) and (27), (29) we will respectively get the following inequalities which will be useful later

$$\left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \leq r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),p(),\alpha(),\Omega}, \; k \in \mathbb{Z}^d, \; r > 0, \; x \in \Omega \tag{30}$$

or

$$\left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega)} \leq r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),p(),\alpha(),\Omega}, \quad x \in \Omega, \quad r > 0, \tag{31}$$

more generally for any cube Q: such that $|Q| < \infty$

$$\| f\chi_Q \|_{L^{q()}(\Omega)} \leq |Q|^{\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),p(),\alpha(),\Omega}, \quad x \in \Omega. \tag{32}$$

We will need the following lemma in functional analysis:

Lemma 17. *Given a normed linear space X, the following statements are equivalent*:

(a) X *is complete*,

(b) $\{x_n\}_n \subset X, \; \sum_n \| x_n \| < \infty \Rightarrow \sum_n x_n < \infty$ *in* X.

3. Properties

Proposition 18. *Let* $\mathbb{Z}^d \subset \Omega$, *given* $q(), \alpha() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d))$

and $f \in L^{q()}_{loc}(\Omega)$. *Then*

$$\left((L^{q()}, l^{p()})^{\alpha()}\ (\Omega), \|\cdot\|_{q(), p(), \alpha()}\right) \textit{ is a Banach space.}$$

Proof. It is clear that $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$ is a sub-vector space of $(L^{q()}, l^{p()})(\Omega)$ and that $f \mapsto \| f \|_{q(), p(), \alpha\, \Omega}$ is a norm on it.

We will use Lemma 17 to prove the desired result:

Let $\{f_n\}_n$ be a sequence in $\left((L^{q()}, l^{p()})^{\alpha()}\ (\Omega), \|\cdot\|_{q(), p(), \alpha(), \Omega}\right)$ such that

$$\sum_{n \in \mathbb{N}} \| f_n \|_{q(), p(), \alpha(), \Omega} < \infty. \tag{33}$$

In other hand, for any $f \in L^{q()}_{loc}(\Omega)$

$$\| f \|_{q(), p(), \alpha(), \Omega} = \sup_{r>0, x \in \Omega} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|_{q(), p(), \Omega},$$

then

$$r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|_{q(), p(), \Omega} \leq \| f \|_{q(), p(), \alpha(), \Omega}, \quad r > 0, \ x \in \Omega. \tag{34}$$

In particular if $r = 1$, we get

$${}_1\| f \|_{q(), p(), \Omega} \leq \| f \|_{q(), p(), \alpha(), \Omega}.$$

But in the precedent Proposition 7-h), we have proved that for any $r > 0$, ${}_1\| \cdot \|_{q(), p(), \Omega}$ and ${}_r\| \cdot \|_{q(), p(), \Omega}$ are equivalent, therefore we get:

$${}_r\| f \|_{q(), p(), \Omega} \leq C \| f \|_{q(), p(), \alpha(), \Omega}. \tag{35}$$

(33) and (35) imply that

$$\sum_{n \in \mathbb{N}^*} {}_r\| f_n \|_{q(), p(), \Omega} < \infty.$$

But $\left((L^{q()}, l^{p()})(\Omega),\ {}_r\|\cdot\|_{q(), p(), \Omega}\right)$ is a Banach space, therefore $\{f_n\}_n$ converges to f in $\left((L^{q()}, l^{p()})(\Omega),\ {}_r\|\cdot\|_{q(), p(), \Omega}\right)$, then it exists an element f of $(L^{q()}, l^{p()})(\Omega)$ such that the series $\sum_{n\in\mathbb{N}} f_n$ converges to f in $(L^{q()}, l^{p()})(\Omega)$, that is, $\sum_{n\in\mathbb{N}} f_n = f$ in $(L^{q()}, l^{p()})(\Omega)$, this implies that

$$ {}_r\left\| \sum_{n\in\mathbb{N}} f_n \right\|_{q(). p(), \Omega} = {}_r\| f \|_{q(), p(), \Omega}, $$

thus

$$ {}_r\| f \|_{q(), p(), \Omega} \le \sum_{n\in\mathbb{N}} {}_r\| f_n \|_{q(), p(), \Omega}, $$

therefore for any $x \in \Omega$:

$$ r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f_n \|_{q(), p(), \Omega} \le \sum_{n\in\mathbb{N}} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f_n \|_{q(), p(), \Omega}. $$

If we pass to the supremum over all $x \in \Omega,\ r > 0,$ we will get:

$$ \| f \|_{q(), p(), \alpha(), \Omega} \le \sum_{n\in\mathbb{N}^*} \| f_n \|_{q(), p(), \alpha(), \Omega} \overset{(33)}{<} \infty, $$

therefore

$$ f \in (L^{q()}, l^{p()})^{\alpha()}(\Omega). $$

Furthermore

$$ \left\| f - \sum_{1\le n\le N} f_n \right\|_{q(), p(), \alpha(), \Omega} \le \sum_{n>N} \| f_n \|_{q(), p(), \alpha(), \Omega} = R_N \qquad \text{and} $$

$\lim_{N\to\infty} R_N = 0,$ since $\sum_{n\in\mathbb{N}^*} \| f_n \|_{q(), p(), \alpha(), \Omega}$ is a convergent series (see (33)).

Therefore the series $\sum_{n\in\mathbb{N}} f_n \to f$ in

$$\left((L^{q()}, l^{p()})^{\alpha()}(\Omega), \|\cdot\|_{q(), p(), \alpha(), \Omega}\right).$$

Proposition 19. *Let* $\mathbb{Z}^d \subset \Omega$, *given* $q(), \alpha() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$ *such that* $q() \leq \alpha() \leq p()$ *and* $f \in L^{q()}_{loc}(\Omega)$, *we suppose that* $q_+ < \infty$ *and* $\frac{1}{q()} - \frac{1}{\alpha()} \in LH_0(\Omega)$. *Then*

$$L^{\alpha()}(\Omega) \subset (L^{q()}, l^{p()})^{\alpha()}(\Omega),$$

that is, there exists a constant C *(eventually equal to* 1) *such that*

$$\| f \|_{q(), p(), \alpha(), \Omega} \leq C \| f \|_{L^{\alpha()}(\Omega)}. \tag{36}$$

Proof. By Hölder's inequality (in variable Lebesgue spaces) we have:

$$\left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} = \left\| \left(f\chi_{I_k^r}\right)\left(\chi_{I_k^r}\right) \right\|_{L^{q()}(\Omega)} \leq C \left\| f\chi_{I_k^r} \right\|_{L^{\alpha()}(\Omega)} \left\| \chi_{I_k^r} \right\|_{L^{\frac{\alpha()q()}{\alpha()-q()}}(\Omega)}. \tag{37}$$

Recall that for any $r() \in \mathcal{P}(\Omega)$ and any measurable $E \subset \Omega$, the harmonic mean r_E of $r()$ on E is defined by:

$$\frac{1}{r_E} = | E |^{-1} \int_E \frac{1}{r(y)} dy$$

and

$$\| \chi_E \|_{L^{r()}(\Omega)} \approx | E |^{\frac{1}{r_E}}$$

otherwise there exists constants c, C such that $c| E |^{\frac{1}{r_E}} \leq \| \chi_E \|_{L^{r()}(\Omega)} \leq C| E |^{\frac{1}{r_E}}$, from this we get

$$\left\| \chi_{I_k^r} \right\|_{L^{\frac{\alpha()q()}{\alpha()-q()}}(\Omega)} \approx \left| I_k^r \right|^{\frac{1}{\left(\frac{\alpha()q()}{\alpha()-q()}\right)_{I_k^r}}},$$

but $\forall y \in \Omega : r_- \leq r(y) \leq r_+$, therefore:

$$\frac{1}{r_+} \leq \frac{1}{r_E} = |E|^{-1}\int_E \frac{1}{r(y)}dy \leq \frac{1}{r_-}.$$

We let $\beta() = \dfrac{1}{\dfrac{\alpha()q()}{\alpha() - q()}} = \dfrac{1}{q()} - \dfrac{1}{\alpha()}$.

$q_+ < \infty \Rightarrow \left|\Omega_\infty^{q()}\right| = 0$, therefore from Remark 2.40 of [47] we have:

$$\min\left\{|I_k^r|^{\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-}, |I_k^r|^{\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}\right\} \leq \left\|\chi_{I_k^r}\right\|_{L^{\beta()}(\Omega)}$$

$$\leq \max\left\{|I_k^r|^{\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-}, |I_k^r|^{\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}\right\},$$

that is,

$$\min\left\{r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-}, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}\right\} \leq \left\|\chi_{I_k^r}\right\|_{L^{\beta()}(\Omega)} \leq \max\left\{r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-}, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}\right\}. \tag{38}$$

Combining (37) and (38), we get

$\left\|f\chi_{I_k^r}\right\|_{L^{q()}(\Omega)} \leq \max\left\{r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-}, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}\right\}\left\|f\chi_{I_k^r}\right\|_{L^{\alpha()}(\Omega)}$, this implies that

$$\left\|\left\{\left\|f\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq \max\left\{r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-}, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}\right\}\left\|\left\{\left\|f\chi_{I_k^r}\right\|_{L^{\alpha()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)},$$

that is,

$$_r\| f \|_{q(),\, p(),\, \Omega} \leq \max\left\{ r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-},\, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \right\} {}_r\| f \|_{\alpha(),\, p(),\, \Omega}$$

$$\overset{\substack{\text{Propo. 7-2-c/-}\bullet \\ p()\geq\alpha()\text{ by hypoth.}}}{\leq} \max\left\{ r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-},\, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \right\} {}_r\| f \|_{\alpha(),\, \alpha(),\, \Omega}$$

$$\overset{\text{Propo. 12}}{=} \max\left\{ r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-},\, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \right\} \| f \|_{L^{\alpha()}(\Omega)},$$

that is,

$$_r\| f \|_{q(),\, p(),\, \Omega} \leq \max\left\{ r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-},\, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \right\} \| f \|_{L^{\alpha()}(\Omega)}. \tag{39}$$

In other hand we have for any $x \in \Omega$,

$$\min\left\{ r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-},\, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \right\} \leq r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \leq \max\left\{ r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-},\, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \right\}.$$

First case: $\min\left\{ r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-},\, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \right\} = r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-}$:

In this case we have

$$\max\left\{ r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-},\, r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \right\} = r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \tag{40}$$

and

$$r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_-} \leq r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \leq r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}. \tag{41}$$

Combining (39) and (40), we will get

$$_r\| f \|_{q(),\, p(),\, \Omega} \leq r^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \| f \|_{L^{\alpha()}(\Omega)}.$$

This gives

$$r^{-d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} r \| f \|_{q(),\, p(),\, \Omega} \leq \| f \|_{L^{\alpha()}(\Omega)}. \tag{42}$$

We have let $\beta() = \dfrac{1}{\dfrac{\alpha()q()}{\alpha() - q()}} = \dfrac{1}{q()} - \dfrac{1}{\alpha()}$, from this, the last numbered inequality becomes $r^{-d\beta_+} r \| f \|_{q(),\, p(),\, \Omega} \leq \| f \|_{L^{\alpha()}(\Omega)}$ which implies that

$$r^{d(\beta(x)-\beta_+)} r^{-d\beta(x)} r \| f \|_{q(),\, p(),\, \Omega} \leq \| f \|_{L^{\alpha()}(\Omega)}. \tag{43}$$

Recall that for any $r() \in \mathcal{P}(\Omega)$

$$r_+ = \operatorname{ess\,sup}_{x\in\Omega} r(x) = \inf\{\varepsilon > 0 : r(x) \leq \varepsilon \ a.e. \ x \in \Omega\}$$

and

$$r_- = \operatorname{ess\,inf}_{x\in\Omega} r(x) = \sup\{\varepsilon > 0 : r(x) \geq \varepsilon \ a.e. \ x \in \Omega\},$$

$$\beta() = \frac{1}{q()} - \frac{1}{\alpha()} \leq \frac{1}{q()},$$

therefore,

$$\left\{\varepsilon > 0 : \frac{1}{q(x)} \leq \varepsilon \ a.e. \ x \in \Omega\right\} \subset \left\{\varepsilon > 0 : \frac{1}{q(x)} - \frac{1}{\alpha(x)} \leq \varepsilon \ a.e. \ x \in \Omega\right\},$$

this implies that

$$\inf\left\{\varepsilon > 0 : \frac{1}{q(x)} - \frac{1}{\alpha(x)} \leq \varepsilon \ a.e. \ x \in \Omega\right\} \leq \inf\left\{\varepsilon > 0 : \frac{1}{q(x)} \leq \varepsilon \ a.e. \ x \in \Omega\right\}$$

which is equivalent to

$$\beta_+ \leq \inf\left\{\varepsilon > 0 : \frac{1}{q(x)} \leq \varepsilon \ a.e. \ x \in \Omega\right\} = \frac{1}{\sup\left\{\frac{1}{\varepsilon} : q(x) \geq \frac{1}{\varepsilon},\ a.e. \ x \in \Omega\right\}} = \frac{1}{q_-} < \infty$$

since $q_- \leq q_+ < \infty$, therefore the hypotheses of Lemma 3.24 of [47] are satisfied, then we can apply it to get:

$$r^{d(\beta(x)-\beta_+)} = \left| J_x^r \right|^{\beta(x)-\beta_+} \leq C(d)$$

and the last numbered inequality becomes:

$$C(d) r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|_{q(), p(), \Omega} \leq \| f \|_{L^{\alpha()}(\Omega)}.$$

Now if we pass to the supremum over all $r > 0$ and $x \in \Omega,$ we will get

$$\| f \|_{q(), p(), \alpha(), \Omega} \leq (C(d))^{-1} \| f \|_{L^{\alpha()}(\Omega)}.$$

Second case: $\min\left\{ r^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_-}, r^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_+} \right\} = r^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_+}$: In this case we have

$$\max\left\{ r^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_-}, r^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_+} \right\} = r^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_-} \tag{44}$$

and

$$r^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_+} \leq r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \leq r^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_-}. \tag{45}$$

Combining (39) and (44), we will get

$${}_r\| f \|_{q(), p(), \Omega} \leq r^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_-} \| f \|_{L^{\alpha()}(\Omega)}.$$

This gives

$$r^{-d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_-} {}_r\| f \|_{q(), p(), \Omega} \leq \| f \|_{L^{\alpha()}(\Omega)}$$

or

$$r^{-d\beta_-} {}_r\| f \|_{q(), p(), \Omega} \leq \| f \|_{L^{\alpha()}(\Omega)}$$

which implies that

$$r^{d(\beta(x) - \beta_-)} r^{-d\beta(x)} {}_r\| f \|_{q(), p(), \Omega} \leq \| f \|_{L^{\alpha()}(\Omega)}. \tag{46}$$

We can apply Lemma 3.24 of [47] to get:

$$r^{d(\beta(x)-\beta_-)} = \left| J_x^r \right|^{\beta(x)-\beta_-} = \left(\left| J_x^r \right|^{\beta_- - \beta(x)} \right)^{-1} \leq K^{-1}(d),$$

where $K(d)$ is a constant only depending on d, and the last numbered inequality becomes:

$$K^{-1}(d) r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|_{q(), p(), \Omega} \leq \| f \|_{L^{\alpha()}(\Omega)}$$

or

$$r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|_{q(), p(), \Omega} \leq K(d) \| f \|_{L^{\alpha()}(\Omega)}.$$

Now if we pass to the supremum over all $r > 0$ and $x \in \Omega$, we will get

$$\| f \|_{q(), p(), \alpha(), \Omega} \leq K(d) \| f \|_{L^{\alpha()}(\Omega)}$$

and the claim is proved.

Proposition 20. *Let* $\mathbb{Z}^d \subset \Omega \subset \mathbb{R}^d$, *given* $q(), \alpha() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$.

(a) *If* $\alpha() < q()$ *on* Ω, *we have* $(L^{q()}, l^{p()})^{\alpha()}(\Omega) = \{0\}$,

(b) *If* $p() < \alpha()$ *on* Ω, *we also have* $(L^{q()}, l^{p()})^{\alpha()}(\Omega) = \{0\}$,

therefore $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$ *is non-trivial if* $q() \leq \alpha() \leq p()$ *on* Ω.

Proof.

(a) $\alpha() < q()$ on Ω

Let $f \in (L^{q()}, l^{p()})^{\alpha()}(\Omega)$, $(\Omega \subset \mathbb{R}^d)$, from (28): $\left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \leq$ ${}_r\| f \|_{q(), p(), \Omega}$.

More generally fix an interval $I \subset \Omega \subset \mathbb{R}^d$ such that $| I | = r^d < \infty$, from the formula just below the formula (29) we have

$$\| f\chi_I \|_{L^{q()}(\Omega)} \leq {}_{|I|^{\frac{1}{d}}}\| f \|_{q(), p(), \Omega}. \tag{47}$$

By definition of $\| f \|_{q(), p(), \alpha(), \Omega}$ for any $x \in \Omega$

$$ {}_r\| f \|_{q(), p(), \Omega} \leq r^{d\left(\frac{1}{q(x)} - \frac{1}{\alpha(x)}\right)} \| f \|_{q(), p(), \alpha(), \Omega},$$

that is,

$$ {}_{|I|^{\frac{1}{d}}}\| f \|_{q(), p(), \Omega} \leq | I |^{\left(\frac{1}{q(x)} - \frac{1}{\alpha(x)}\right)} \| f \|_{q(), p(), \alpha(), \Omega}. \tag{48}$$

These two last numbered inequalities give:

$$\| f\chi_I \|_{L^{q()}(\Omega)} \leq | I |^{\left(\frac{1}{q(x)} - \frac{1}{\alpha(x)}\right)} \| f \|_{q(), p(), \alpha(), \Omega},$$

this is equivalent to:

$$\| f\chi_I \|_{L^{q()}(\Omega)} \leq \frac{\| f \|_{q(), p(), \alpha(), \Omega}}{| I |^{\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)}},$$

$f \in (L^{q()}, l^{p()})^{\alpha()}(\Omega) \Rightarrow \| f \|_{q(), p(), \alpha(), \Omega} < \infty,$ $\alpha() < q() \Rightarrow \frac{1}{\alpha(x)} - \frac{1}{q(x)}$ > 0 on Ω.

If we tend $I \to \mathbb{R}^d$ (then $r^d = | I | \to \infty$), we will get $\| f \|_{L^{q()}(\Omega)} = 0$, therefore $f = 0$.

(b) Let $f \in (L^{q()}, l^{p()})^{\alpha()}(\Omega)$. $p() < \alpha()$. Fix $x \in \Omega$.

$$\| fg \|_{L^1(\Omega)} \overset{\text{Propo. 12}}{=} {}_r\| fg \|_{1,1,\Omega}$$

$$\overset{\text{Propo. 7-2-f}}{\leq} {}_r\| f \|_{q(), p(), \Omega} \times {}_r\| g \|_{q'(), p'(), \Omega}$$

$$\overset{(27)}{\leq} r^{d\left(\frac{1}{q(x)} - \frac{1}{\alpha(x)}\right)} \| f \|_{q(), p(), \alpha(), \Omega} \times r^{d\left(\frac{1}{q'(x)} - \frac{1}{p'(x)}\right)} \| g \|_{q'(), p'(), p'(), \Omega}$$

$$= r^{d\left(\frac{1}{q(x)} - \frac{1}{\alpha x} + \frac{1}{q'(x)} - \frac{1}{p'(x)}\right)} \| f \|_{q(), p(), \alpha(), \Omega} \times \| g \|_{q'(), p'(), p'(), \Omega}$$

$$= r^{d\left(1 - \frac{1}{\alpha(x)} - \frac{1}{p'(x)}\right)} \| f \|_{q(), p(), \alpha(), \Omega} \times \| g \|_{q'(), p'(), p'(), \Omega}$$

$$\overset{\text{Propo.19}}{\leq} r^{d\left(\frac{1}{p(x)} - \frac{1}{\alpha(x)}\right)} \| f \|_{q(), p(), \alpha(), \Omega} \times \| g \|_{L^{p'()}(\Omega)},$$

that is

$$\| fg \|_{L^1(\Omega)} \leq r^{d\left(\frac{1}{p(x)} - \frac{1}{\alpha(x)}\right)} \| f \|_{q(), p(), \alpha(), \Omega} \times \| g \|_{L^{p'()}(\Omega)}. \tag{49}$$

Since $p() < \alpha()$ and $f \in (L^{q()}, l^{p()})^{\alpha()}(\Omega)$, we have $\frac{1}{p(x)} - \frac{1}{\alpha(x)} > 0$ and $\| f \|_{q(), p(), \alpha(), \Omega} = C < \infty$, therefore we get

$$\| fg \|_{L^1(\Omega)} \leq Cr^{d\left(\frac{1}{p(x)} - \frac{1}{\alpha(x)}\right)} \| g \|_{L^{p'()}(\Omega)}.$$

Let us define the function g on Ω by $g(x) = \chi_{J_0^1}(x)$, in this condition from Lemma 2.39 of [47] $\| g \|_{L^{p'()}(\Omega)} \leq 2$, thus we have

$$\| fg \|_{L^1(\Omega)} \leq 2Cr^{d\left(\frac{1}{p(x)} - \frac{1}{\alpha(x)}\right)},$$

since $\frac{1}{p(x)} - \frac{1}{\alpha(x)} > 0$, now if we tend r to zero, we will get $\| fg \|_{L^1(\Omega)} = 0$, since $g \neq 0$ on Ω, we have $f = 0$.

Therefore $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$ is non-trivial if $q() \leq \alpha() \leq p()$ on Ω containing $\mathbb{Z}^d$, in the sequel we will only consider $q() \leq \alpha() \leq p()$.

Proposition 21. *Let* $\mathbb{Z}^d \subset \Omega \subset \mathbb{R}^d$, *given* $q(), \alpha() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$ *such that* $q() \leq \alpha() \leq p()$ *on* Ω *and* $f \in (L^{q()}, l^{p()})^{\alpha()}(\Omega)$.

If $\max\left\{\frac{1}{q_-}, \frac{1}{p_-}, \frac{1}{\alpha_-}\right\} \leq s < \infty$, $\left|\Omega_\infty^{q()}\right| = 0$, *then*

$$\left\| |f|^s \right\|_{q(), p(), \alpha(), \Omega} = \| f \|^s{}_{sq(), sp(), s\alpha(), \Omega}.$$

Proof. Under the hypotheses of the proposition, from Proposition 7-2)-a), we have

$$_r\left\| |f|^s \right\|_{q(), p(), \Omega} = {}_r\| f \|^s{}_{sq(), sp(), \Omega},$$

then for $x \in \Omega$, we get:

$$r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\left\| |f|^s \right\|_{q(), p(), \Omega} = r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|^s{}_{sq(), sp(), \Omega},$$

therefore

$$r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\left\| |f|^s \right\|_{q(), p(), \Omega} = \left(r^{d\left(\frac{1}{s\alpha(x)} - \frac{1}{sq(x)}\right)} {}_r\| f \|_{sq(), sp(), \Omega} \right)^s.$$

If we pass to the supremum over all $x \in \Omega$, $r > 0$, we will get:

$$\left\| |f|^s \right\|_{q(), p(), \alpha(), \Omega} = \| f \|_{sq(), sp(), s\alpha(), \Omega}.$$

Remark 22. If $q() \leq \alpha() \leq p()$ on Ω and $\begin{cases} q() = q = \text{constant real} \\ p() = p = \text{constant real} \\ \alpha() = \alpha = \text{constant real} \end{cases}$ all in $[1, \infty]$, then $(L^{q()}, l^{p()})^{\alpha()}(\Omega) = (L^q, l^p)^\alpha(\Omega)$ with constant exponents.

The space $(L^q, l^p)^\alpha(\Omega)$ with constant exponents have been widely studied during these last twenty years (See [13, 14,15, 25, 17]).

Indeed if $\begin{cases} q() = q = \text{constant real} \\ p() = p = \text{constant real} \\ \alpha() = \alpha = \text{constant real} \end{cases}$ all in $[1, \infty]$, then

$$\| f \|_{q(), p(), \alpha(), \Omega} = \sup_{r>0, x\in\Omega} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|_{q(), p(), \Omega},$$

we have seen in a precedent work in [22] that when $\begin{cases} q() = q = \text{constant real} \\ p() = p = \text{constant real} \end{cases}$ all in $[1, \infty]$, then ${}_r\| f \|_{q(), p(), \Omega} = {}_r\| f \|_{q, p, \Omega}$, that is $(L^{q()}, l^{p()})(\Omega)$ defined by ${}_r\| f \|_{q(), p(), \Omega}$ coincides with $(L^q, l^p)(\Omega)$ given precedently, thus

$$\| f \|_{q(), p(), \alpha(), \Omega} = \sup_{r>0} r^{d\left(\frac{1}{\alpha} - \frac{1}{q}\right)} {}_r\| f \|_{q, p, \Omega} = \| f \|_{q, p, \alpha, \Omega}.$$

We conclude that the spaces $L^{q()}, l^{p()})^{\alpha()}(\Omega)$ generalize $(L^q, l^p)^{\alpha}(\Omega)$ which are studied by many researchers, see [7-12].

Proposition 23. *Let* $\mathbb{Z}^d \subset \Omega \subset \mathbb{R}^d$, *given* $q_1(), q_2(), \alpha() \in \mathcal{P}(\Omega)$, $p_1(), p_2() \in \mathcal{P}(\mathbb{Z}^d)$ *and* $f \in L^{q()}_{loc}(\Omega)$.

(1) *Suppose that* $q_1() \le q_2() \le \alpha() \le p()$, $\left| \Omega_\infty^{q_1()} \right| = 0$, $\frac{1}{q_3()} = \frac{1}{q_1()} - \frac{1}{q_2()} \in LH_0(\Omega)$, $(q_3)_+ = q_{3_+} < \infty$, *and* $f \in (L^{q_2()}, l^{p()})^{\alpha()}$.

Then there exists a constant C *such that*

$$\| f \|_{q_1(), p(), \alpha(), \Omega} \le C \| f \|_{q_2(), p(), \alpha(), \Omega},$$

in other words under these hypotheses we have

$$(L^{q_2()}, l^{p()})^{\alpha()}(\Omega) \subset (L^{q_1()}, l^{p()})^{\alpha()}(\Omega).$$

(2) *Suppose that* $q() \le \alpha() \le p_1() \le p_2()$ *and* $f \in (L^{q()}, l^{p_1()})^{\alpha()}(\Omega)$. *Then*

$$\| f \|_{q(), p_2(), \alpha(), \Omega} \le \| f \|_{q(), p_1(), \alpha(), \Omega}.$$

In other words under the hypotheses we have the following embedding

$$(L^{q()},\, l^{p_1()})^{\alpha()}(\Omega) \subset (L^{q()},\, l^{p_2()})^{\alpha()}(\Omega).$$

Proof. (1) $q_1() \leq q_2() \Rightarrow \dfrac{1}{q_1()} \geq \dfrac{1}{q_2()}$, therefore there exists a function $q_3() \in \mathcal{P}(\Omega)$ such that $\dfrac{1}{q_1()} = \dfrac{1}{q_2()} + \dfrac{1}{q_3()}$. By Hölder's inequality in variable Lebesgue spaces, there exists a constant C such that

$$\left\| f\chi_{I_k^r} \right\|_{L^{q_1()}(\Omega)} = \left\| \left(f\chi_{I_k^r}\right)\chi_{I_k^r} \right\|_{L^{q_1()}(\Omega)} \leq C \cdot \left\| f\chi_{I_k^r} \right\|_{L^{q_2()}(\Omega)} \left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)},$$

that is

$$\left\| f\chi_{I_k^r} \right\|_{L^{q_1()}(\Omega)} \leq C \cdot \left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)} \cdot \left\| f\chi_{I_k^r} \right\|_{L^{q_2()}(\Omega)}. \tag{50}$$

By Remark 2.40 of [47], we have

$$\min\left\{ \left| I_k^r \right|^{\frac{1}{q_{3-}}},\, \left| I_k^r \right|^{\frac{1}{q_{3+}}} \right\} \leq \left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)} \leq \max\left\{ \left| I_k^r \right|^{\frac{1}{q_{3-}}},\, \left| I_k^r \right|^{\frac{1}{q_{3+}}} \right\},$$

that is,

$$\min\left\{ r^{\frac{d}{q_{3-}}},\, r^{\frac{d}{q_{3+}}} \right\} \leq \left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)} \leq \max\left\{ r^{\frac{d}{q_{3-}}},\, r^{\frac{d}{q_{3+}}} \right\}. \tag{51}$$

From this last inequality, we remark that $\left\| \chi_{I_k^r} \right\|_{L^{q_1()}(\Omega)}$ is (upper-) bounded (and lower-bounded) by a constant which does not depend on k, therefore (50) implies that

$$\left\| \left\{ \left\| f\chi_{I_k^r} \right\|_{L^{q_1()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \leq C \left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)} \left\| \left\{ \left\| f\chi_{I_k^r} \right\|_{L^{q_2()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)},$$

that is,

$${}_r\| f \|_{q_1(), p(), \Omega} \leq C \left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)} {}_r\| f \|_{q_2(), p(), \Omega},$$

multiplying this inequality by $r^{\frac{d}{\alpha(x)}}$, $(x \in \Omega)$, we get

$$r^{\frac{d}{q_1(x)}} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q_1(x)}\right)} {}_r\| f \|_{q_1(), p(), \Omega}$$

$$\leq C \left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)} r^{\frac{d}{q_2(x)}} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q_2(x)}\right)} {}_r\| f \|_{q_2(), p(), \Omega}. \tag{52}$$

This leads to

$$r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q_1(x)}\right)} {}_r\| f \|_{q_1(), p(), \Omega}$$

$$\leq C \left(\left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)} r^{d\left(\frac{1}{q_2(x)} - \frac{1}{q_1(x)}\right)} \right) r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q_2(x)}\right)} {}_r\| f \|_{q_2(), p(), \Omega}. \tag{53}$$

It remains to bound $\left[\left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)} r^{d\left(\frac{1}{q_2(x)} - \frac{1}{q_1(x)}\right)} \right]$ by a constant K which does not depend on r. Recall that $\frac{1}{q_3()} = \frac{1}{q_1()} - \frac{1}{q_2()}$ on Ω, therefore

$$\begin{aligned} \left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)} r^{d\left(\frac{1}{q_2(x)} - \frac{1}{q_1(x)}\right)} &\overset{(51)}{\leq} \max\left\{ r^{\frac{d}{q_{3-}}}, r^{\frac{d}{q_{3+}}} \right\} r^{d\left(\frac{1}{q_2(x)} - \frac{1}{q_1(x)}\right)} \\ &= \max\left\{ r^{\frac{d}{q_{3-}}}, r^{\frac{d}{q_{3+}}} \right\} r^{-d\left(\frac{1}{q_1(x)} - \frac{1}{q_2(x)}\right)} \\ &= \max\left\{ r^{\frac{d}{q_{3-}}}, r^{\frac{d}{q_{3+}}} \right\} r^{-\frac{d}{q_3()}} \\ &= \max\left\{ r^{d\left(\frac{1}{q_{3-}} - \frac{1}{q_3(x)}\right)}, r^{d\left(\frac{1}{q_{3+}} - \frac{1}{q_3(x)}\right)} \right\} \end{aligned}$$

$$= \max\left\{ \left| J_x^r \right|^{\frac{1}{q_{3_-}} - \frac{1}{q_3(x)}}, \left| J_x^r \right|^{\frac{1}{q_{3_+}} - \frac{1}{q_3(x)}} \right\}.$$

Recall that since $q_{3_+} < \infty : q_3() \in LH_0(\Omega) \Leftrightarrow \frac{1}{q_3()} \in LH_0(\Omega)$.

$$\left| J_x^r \right|^{\frac{1}{q_{3_-}} - \frac{1}{q_3(x)}} = \left(\left| J_x^r \right|^{\frac{q_{3_-} - q_3(x)}{q_3(x) q_{3_+}}} \right)^{-1}$$

$$= \left(\left| J_x^r \right|^{q_{3_-} - q_3(x)} \right)^{\frac{-1}{q_3(x) q_{3_+}}}$$

$$\leq \max\left\{ \left(\left| J_x^r \right|^{q_{3_-} - q_3(x)} \right)^{\frac{-1}{q_{3_-} q_{3_+}}}, \left(\left| J_x^r \right|^{q_{3_-} - q_3(x)} \right)^{\frac{-1}{(q_{3_+})^2}} \right\}$$

$$\overset{\text{Lem 3.24 of [48]}}{\leq} \max\left\{ (C(d))^{\frac{-1}{q_{3_-} q_{3_+}}}, (C(d))^{\frac{-1}{(q_{3_+})^2}} \right\}$$

$$= K_1(d) < \infty.$$

By the same way we prove that $\left| J_x^r \right|^{\frac{1}{q_{3_+}} - \frac{1}{q_3(x)}} \leq K_2(d) < \infty$ if we let $\max\{K_1(d), K_2(d)\} = K(d)$, therefore in any case $\left\| \chi_{I_k^r} \right\|_{L^{q_3()}(\Omega)} \times r^{d\left(\frac{1}{q_2(x)} - \frac{1}{q_1(x)}\right)} \leq K(d)$, combining this result with the last numbered inequality (53) we will get

$$r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q_1(x)}\right)} {}_r\| f \|_{q_1(), p(), \Omega} \leq CK(d) r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q_2(x)}\right)} {}_r\| f \|_{q_2(), p(), \Omega}.$$

Now if we pass to supremum over all $r > 0$, $x \in \Omega$, we will get

$$\| f \|_{q_1(), p(), \alpha(), \Omega} \leq CK(d) \| f \|_{q_2(), p(), \alpha(), \Omega}.$$

(2) Under the hypotheses of Proposition 23, from (8) we have:

$$\left\| \left\{ \| f\chi_{I_k^r} \|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p_2()}(\mathbb{Z}^d)} \leq \left\| \left\{ \| f\chi_{I_k^r} \|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p_1()}(\mathbb{Z}^d)}.$$

This implies that:

$$r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \left\| \left\{ \| f\chi_{I_k^r} \|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p_2()}(\mathbb{Z}^d)}$$

$$\leq r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \left\| \left\{ \| f\chi_{I_k^r} \|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p_1()}(\mathbb{Z}^d)}.$$

If we pass to the supremum over all $x \in \Omega$, $r > 0$, we will get:

$$\| f \|_{q(), p_2(), \alpha(), \Omega} \leq \| f \|_{q(), p_1(), \alpha(), \Omega}.$$

Proposition 24. *Let* $\mathbb{Z}^d \subset \Omega \subset \mathbb{R}^d$, *suppose that*:

• $q(), \alpha(), q_1(), \alpha_1(), q_2(), \alpha_2() \in \mathcal{P}(\Omega)$, $p(), p_1(), p_2() \in \mathcal{P}(\mathbb{Z}^d)$, *such that* $q_1() \leq \alpha_1() \leq p()$, $q_2() \leq \alpha_2() \leq p()$, $q_3() \leq \alpha_3() \leq p()$ *on* Ω.

• $$\begin{cases} \dfrac{1}{q_1()} + \dfrac{1}{q_2()} = \dfrac{1}{q()}, \\ \dfrac{1}{p_1()} + \dfrac{1}{p_2()} = \dfrac{1}{p()}, \\ \dfrac{1}{\alpha_1()} + \dfrac{1}{\alpha_2()} = \dfrac{1}{\alpha()}. \end{cases}$$

Then there exists a positive constant K *such that for any* $(f, g) \in (L^{q_1()}, l^{p_1()})^{\alpha_1()}(\Omega) \times (L^{q_2()}, l^{p_2()})^{\alpha_2()}(\Omega)$:

$$\| fg \|_{q(), p(), \alpha(), \Omega} \leq K \| f \|_{q_1(), p_1(), \alpha_1(), \Omega} \| g \|_{q_2(), p_2(), \alpha_2(), \Omega}.$$

Proof. By Hölder's inequality (in variable Lebesgue spaces), we have:

$$\left\| fg\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \leq K \left\| f\chi_{I_k^r} \right\|_{L^{q_1()}(\Omega)} \left\| g\chi_{I_k^r} \right\|_{L^{q_2()}(\Omega)}.$$

$\| \cdot \|_{l^{p()}(\mathbb{Z}^d)}$ is order preserving, then

$$\left\|\left\{\left\|fg\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq K\left\|\left\{\left\|f\chi_{I_k^r}\right\|_{L^{q_1()}(\Omega)}\left\|g\chi_{I_k^r}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_1()}(\mathbb{Z}^d)}.$$

Again by Hölder's inequality, the second member of the last inequality is bounded by the following inequality's second member:

$$\left\|\left\{\left\|fg\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq K\left\|\left\{\left\|f\chi_{I_k^r}\right\|_{L^{q_1()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_1()}(\mathbb{Z}^d)}\left\|\left\{\left\|g\chi_{I_k^r}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_2()}(\mathbb{Z}^d)},$$

this implies that:

$$r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)}\left\|\left\{\left\|fg\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq Kr^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)}\left\|\left\{\left\|f\chi_{I_k^r}\right\|_{L^{q_1()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_1()}(\mathbb{Z}^d)}\left\|\left\{\left\|g\chi_{I_k^r}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_2()}(\mathbb{Z}^d)},$$

therefore

$$r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)}\left\|\left\{\left\|fg\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq Kr^{d\left(\frac{1}{\alpha_1(x)}-\frac{1}{q_1(x)}\right)}\left\|\left\{\left\|f\chi_{I_k^r}\right\|_{L^{q_1()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_1()}(\mathbb{Z}^d)}$$

$$\times r^{d\left(\frac{1}{\alpha_2(x)}-\frac{1}{q_2(x)}\right)}\left\|\left\{\left\|g\chi_{I_k^r}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_2()}(\mathbb{Z}^d)}.$$

If we pass to the supremum over all the $r > 0,\ x \in \mathbb{R}^d,$ we will get:

$$\| fg \|_{q(), p(), \alpha(), \Omega} \leq K \| f \|_{q_1(), p_1(), \alpha_1(), \Omega} \| g \|_{q_2(), p_2(), \alpha_2(), \Omega}.$$

Proposition 25. *Let* $\mathbb{Z}^d \subset \Omega$, *given* $q(), \alpha() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$ *such that* $q() = \alpha() \leq p()$. *Then*

$$(L^{q()}, l^{p()})^{q()}(\Omega) = L^{q()}(\Omega),$$

that is, there exists positive real numbers c, C *such that*

$$c\| f \|_{L^{q()}(\Omega)} \leq \| f \|_{q(), p(), q(), \Omega} \leq C \| f \|_{L^{q()}(\Omega)}. \tag{54}$$

Proof. By the Proposition 19: $L^{q()}(\Omega) \subset (L^{q()}, l^{p()})^{q()}(\Omega)$, it remains to prove that $(L^{q()}, l^{p()})^{q()}(\Omega) \subset L^{q()}(\Omega)$, that is, there exists a positive constant K such that

$$\| f \|_{L^{q()}(\Omega)} \leq K \cdot \| f \|_{q(), p(), q(), \Omega}.$$

Let $f \in (L^{q()}, l^{p()})^{q()}(\Omega)$, then we have:

$$0 \leq M = \| f \|_{q(), p(), q(), \Omega} = \sup_{r>0, x\in\Omega} r^{d\left(\frac{1}{q(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|_{q(), p(), \Omega}$$

$$= \sup_{r>0} {}_r\| f \|_{q(), p(), \Omega} < \infty.$$

By definition of $\| f \|_{q(), p(), \alpha(), \Omega}$, we have that $r^{d\left(\frac{1}{q(x)} - \frac{1}{q(x)}\right)} {}_r\| f \|_{q(), p(), \Omega} \leq \| f \|_{q(), p(), q(), \Omega}$, $r > 0$, $x \in \Omega$, that is,

$${}_r\| f \|_{q(), p(), \Omega} \leq \| f \|_{q(), p(), q(), \Omega} = M. \tag{55}$$

Using (55) and (28), we will get

$$\left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \leq \| f \|_{q(), p(), q(), \Omega} = M. \tag{56}$$

Remark that $[-r, r[^d = \bigcup_{k\in\{-1,0\}^d} I_k^r$, $r > 0$ and $\{-1, 0\}^d$ owns 2^d

members, thus

$$\left\| f\chi_{[-r,\, r[^d} \right\|_{L^{q()}(\Omega)} = \left\| \sum_{k\in\{-1,0\}^d} f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)}$$

$$\leq \sum_{k\in\{-1,0\}^d} \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \overset{(56)}{\leq} \sum_{k\in\{-1,0\}^d} M = 2^d M, \quad r > 0,$$

therefore $\left\| f\chi_{[-n,\, n[^d} \right\|_{L^{q()}(\Omega)} \leq 2^d MC,\ \ n \in \mathbb{N},$ but $\left(|f|^{q()}\ \chi_{[-n,\, n[^d} \right) \uparrow |f|^{q()},$ from convergence monotone Theorem 8 we have:

$$\| f \|_{L^{q()}(\Omega)} = \lim_{n\to\infty} \left\| f\chi_{[-n,\, n[^d} \right\|_{L^{q()}(\Omega)} \leq 2^d M = 2^d \| f \|_{q(),\, p(),\, q(),\, \Omega}$$

and the claim is proved.

Proposition 28. *Let* $\mathbb{Z}^d \subset \Omega,$ *given* $q(), \alpha() \in \mathcal{P}(\Omega),$ $p() \in \mathcal{P}(\mathbb{Z}^d)$ *such that* $q() \leq \alpha() = p().$ *Then*

$$(L^{q()},\, l^{p()})^{p()}(\Omega) = L^{p()}(\Omega),$$

that is, there exists positive real numbers c and C *such that*

$$c\| f \|_{L^{p()}(\Omega)} \leq \| f \|_{q(),\, p(),\, p(),\, \Omega} \leq C\| f \|_{L^{p()}(\Omega)}. \tag{57}$$

Proof. From Proposition 19 we have

$$L^{p()}(\Omega) \subset (L^{q()},\, l^{p()})^{p()}(\Omega).$$

It remains to show that $(L^{q()},\, l^{p()})^{p()}(\Omega) \subset L^{p()}(\Omega),$ that is, there exists a positive constant K such that

$$\| f \|_{L^{p()}(\Omega)} \leq K\| f \|_{q(),\, p(),\, p(),\, \Omega}.$$

Let $f \in (L^{q()},\, l^{p()})^{p()}(\Omega).$

Consider an element g of $L^{p'()}(\Omega).$

Recall that $\left\| \chi_{I_k^r} \right\|_{L^{q'()}(\Omega)} \approx \left| I_k^r \right|^{\frac{1}{q'_{I_k^r}}}$, that is $\exists l,\ L > 0$:

$$l \left| I_k^r \right|^{\frac{1}{q'_{I_k^r}}} \leq \left\| \chi_{I_k^r} \right\|_{L^{q'()}(\Omega)} \leq L \left| I_k^r \right|^{\frac{1}{q'_{I_k^r}}}, \tag{58}$$

where $q'_{I_k^r}$ is the harmonic mean of $q'()$ on I_k^r defined on Ω by $\frac{1}{q'_{I_k^r}} = \left| I_k^r \right|^{-1} \int_{I_k^r} \frac{1}{q'(y)} dy.$

• Suppose that g is a simple function under the form: $g = \sum_{k \in A} a_k \chi_{I_k^r}$ where A is a finite subset of $\mathbb{Z}^d$ and $0 < r < \infty$.

$$\| fg \|_{L^1(\Omega)} \leq \sum_{k \in A} | a_k | \left\| f \chi_{I_k^r} \right\|_{L^1(\Omega)}$$

$$= \sum_{k \in A} | a_k | \left\| \left(f \chi_{I_k^r} \right) \chi_{I_k^r} \right\|_{L^1(\Omega)}$$

$$\overset{\text{Holder in. } L^{q()}}{\leq} C \sum_{k \in A} | a_k | \left\| f \chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \left\| \chi_{I_k^r} \right\|_{L^{q'()}(\Omega)}$$

$$\overset{(58)}{\leq} CL \sum_{k \in A} | a_k | \left| I_k^r \right|^{\frac{1}{q'_{I_k^r}}} \left\| f \chi_{I_k^r} \right\|_{L^{q()}(\Omega)}$$

$$= C \sum_{k \in A} | a_k | \, r^{\frac{d}{q'_{I_k^r}}} \left\| f \chi_{I_k^r} \right\|_{L^{q()}(\Omega)}$$

$$= CL \sum_{k \in A} | a_k | \, r^{d \left(1 - \frac{1}{q_{I_k^r}} \right)} \left\| f \chi_{I_k^r} \right\|_{L^{q()}(\Omega)}$$

$$\overset{\text{Holder in. } l^{p()}}{\leq} CL\| \{a_k\}_{k\in A} \|_{l^{p'()}(A)} r^{d\left(1-\frac{1}{q_{I_k^r}}\right)} \left\| \left\{ \| f\chi_{I_k^r} \|_{L^{q()}(\Omega)} \right\}_{k\in A} \right\|_{l^{p()}(A)}$$

$$= CL\| \{a_k\}_{k\in A} \|_{l^{p'()}(A)} r^{\frac{d}{p'_{I_k^r}}} \times r^{d\left(\frac{1}{p_{I_k^r}}-\frac{1}{q_{I_k^r}}\right)} \left\| \left\{ \| f\chi_{I_k^r} \|_{L^{q()}(\Omega)} \right\}_{k\in A} \right\|_{l^{p()}(A)}$$

$$\leq CL\| \{a_k\}_{k\in A} \|_{l^{p'()}(A)} r^{\frac{d}{p'_{I_k^r}}} \times r^{d\left(\frac{1}{p_{I_k^r}}-\frac{1}{q_{I_k^r}}\right)} \left\| \left\{ \| f\chi_{I_k^r} \|_{L^{q()}(\Omega)} \right\}_{k\in \Omega} \right\|_{l^{p()}(\Omega)}$$

$$\leq CL\| \{a_k\}_{k\in A} \|_{l^{p'()}(A)} r^{\frac{d}{p'_{I_k^r}}} \| f \|_{q(), p(), p(), \Omega}$$

$$= CL\| \{a_k\}_{k\in A} \|_{l^{p'()}(A)} | I_k^r |^{\frac{1}{p'_{I_k^r}}} \| f \|_{q(), p(), p(), \Omega}$$

$$\overset{(8)}{\leq} CL\| \{a_k\}_{k\in A} \|_{l^{1}(A)} | I_k^r |^{\frac{1}{p'_{I_k^r}}} \| f \|_{q(), p(), p(), \Omega}$$

$$\overset{(58)}{\leq} CL l^{-1}\| \{a_k\}_{k\in A} \|_{l^{1}(A)} \left\| \chi_{I_k^r} \right\|_{L^{p'()}(A)} \| f \|_{q(), p(), p(), \Omega}$$

$$\overset{\text{def. of } \|\cdot\|_{l^1}}{=} CL l^{-1} \sum_{k\in A} | a_k | \left\| \chi_{I_k^r} \right\|_{L^{p'()}(A)} \| f \|_{q(), p(), p(), \Omega}$$

$$\overset{\text{proper of } \|\cdot\|_{L^{p'()}}}{=} CL l^{-1} \sum_{k\in A} \left\| a_k \chi_{I_k^r} \right\|_{L^{p'()}(A)} \| f \|_{q(), p(), p(), \Omega}$$

$$\overset{\text{cons. of Lem. 11}}{=} CL l^{-1} \left\| \sum_{k\in A} a_k \chi_{I_k^r} \right\|_{L^{p'()}(A)} \| f \|_{q(), p(), p(), \Omega}$$

$$\leq CL l^{-1} \| g \|_{L^{p'()}(\Omega)} \| f \|_{q(), p(), p(), \Omega}.$$

That is,

$$\| fg \|_{L^1(\Omega)} \leq CLl^{-1} \| g \|_{L^{p'()}(\Omega)} \| f \|_{q(), p(), p(), \Omega}.$$

• Since the set of simple functions are dense in $L^{p'()}(\Omega)$, we have the following

$$\| fg \|_{L^1(\Omega)} \leq CLl^{-1} \| g \|_{L^{p'()}(\Omega)} \| f \|_{q(), p(), p(), \Omega}, \quad g \in L^{p'()}(\Omega). \tag{59}$$

Therefore we have:

$$\| f \|_{L^{p()}(\Omega)} = \sup \{ \| fg \|_{L^1(\Omega)} : g \in L^{p'()}(\Omega), \| g \|_{L^{p'()}(\Omega)} \leq 1 \}$$

$$\leq CLl^{-1} \sup \{ \| g \|_{L^{p'()}(\Omega)} \| f \|_{q(), p(), p(), \Omega} : g \in L^{p'()}(\Omega), \| g \|_{L^{p'()}(\Omega)} \leq 1 \}$$

$$\leq CLl^{-1} \sup \{ \| f \|_{q(), p(), p(), \Omega} : g \in L^{p'()}(\Omega), \| g \|_{L^{p'()}(\Omega)} \leq 1 \},$$

that is,

$$\| f \|_{L^{p()}(\Omega)} \leq CLl^{-1} \| f \|_{q(), p(), p(), \Omega}$$

which means that

$$(L^{q()}, l^{p()})^{p()}(\Omega) \subset L^{p()}(\Omega)$$

and the claim is proved.

Definition 27. Let $\mathbb{Z}^d \subset \Omega$, given $q() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$, we define the function $\| \cdot \|_{q(), p(), \Omega}$ on $(L^{q()}, l^{p()})(\Omega)$ by

$$\| f \|_{q(), p(), \Omega} = \begin{cases} \left\| \min\left\{ r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}} \right\} \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} & \text{if } p_+ < \infty, \\ \sup_{k \in \mathbb{Z}^d} \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega, dy)} & \text{if } p_+ = \infty, \end{cases} \tag{60}$$

we have taken account of (3) of Remark 9, that is, the zero extension of $p()$ on $\Omega \setminus \mathbb{Z}^d$.

Remark 28. We fix $q() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$, given $f \in (L^{q()}, l^{p()})(\Omega)$ and take account of the zero extension of $p()$ on $\Omega \setminus \mathbb{Z}^d$ (of Remark 9), to compute the real number $\left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$, we first calculate $\left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)}$:

$$\left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} = \inf\left\{\lambda > 0 : \rho_{L^{q()}(\Omega, dy)}\left(\frac{f\chi_{J_x^r}}{\lambda}\right) \le 1\right\},$$

(where $\rho_{L^{q()}(\Omega, dy)}\left(\frac{f\chi_{J_x^r}}{\lambda}\right) = \int_{\Omega \setminus \Omega_\infty^{q()}} \left| \frac{f(y)\chi_{J_x^r}}{\lambda} \right|^{q(y)} dy + \left\| \frac{f\chi_{J_x^r}}{\lambda} \right\|_{L^\infty(\Omega_\infty^{q()})}$), x is supposed to be a real parameter, the result of the calculation (of $\left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)}$) depends on $x \in \Omega$ and $r > 0$, we denote it by $\beta(x, r)$ and consider the function $x \mapsto \beta(x, r)$, after that, we determine $\left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$ by:

$$\left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} = \| \beta(\cdot, r) \|_{L^{p()}(\Omega, dx)}$$

$$= \inf\left\{\lambda > 0 : \rho_{L^{p()}(\Omega, dx)}\left(\frac{\beta(\cdot, r)}{\lambda}\right) \le 1\right\},$$

where

$$\rho_{L^{p()}(\Omega, dx)}\left(\frac{\beta(\cdot, r)}{\lambda}\right) = \int_{\Omega \setminus \Omega_\infty^{p()}} \left| \frac{\beta(x, r)}{\lambda} \right|^{p(x)} dx + \left\| \frac{\beta(\cdot, r)}{\lambda} \right\|_{L^\infty(\Omega_\infty^{p()})}.$$

The result of the calculation of $\left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$ depends on r.

Proposition 29. *Let* $\mathbb{Z}^d \subset \Omega$, $q() \in \mathcal{P}(\Omega)$ *such that* $q_+ < \infty$ *and*

$p() \in \mathcal{P}(\mathbb{Z}^d)$. *Then there exist real numbers* A *and* B *such that*:

$$A_r \| f \|_{q(), p(), \Omega} \leq \||f \||_{q(), p(), \Omega} \leq B_r \| f \|_{q(), p(), \Omega}.$$

That is, ${}_r\| \cdot \|_{q(), p(), \Omega}$ and $\||\cdot\||_{q(), p(), \Omega}$ are equivalent on $(L^{q()}, l^{p()})(\Omega)$.

Proof. It is already known that ${}_r\| \cdot \|_{q(), p(), \Omega}$ is a norm on $(L^{q()}, l^{p()})(\Omega)$ (see [22]). Furthermore, according to its definition, it is easy to see that $\||\cdot\||_{q(), p(), \Omega}$ is a norm on $(L^{q()}, l^{p()})(\Omega)$.

Case 1: $p_+ < \infty$.

Let $f \in L^q_{loc}(\Omega)$.

(a) Remark that for any $(x, k) \in \Omega \times \mathbb{Z}^d$, we have from Lemma 5-(b) that

$$x \in I_k^r \Rightarrow J_x^r \subset \bigcup_{l \in L_k} I_l^r, \tag{e_1}$$

where $L_k = \{l \in \mathbb{Z}^d : k_j - 1 \leq l_j \leq k_j + 1\}$ for $j \in \{1, ..., d\}$.

By consequence

$$\left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\overset{(e_1)}{\leq} \left\| \left\| f\chi_{\bigcup_{l \in L_k} I_l^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\overset{\text{Lem. 10-(ii)}}{\leq} \left\| \sum_{l \in L_k} \| f \|_{L^{q()}(I_l^r, dy)} \right\|_{L^{p()}\left(\bigcup_{k \in \mathbb{Z}^d} I_k^r, dx\right)}$$

$$= \sum_{k \in \mathbb{Z}^d} \left\| \sum_{l \in L_k} \| f \|_{L^{q()}(I_l^r, dy)} \right\|_{L^{p()}(I_k^r, dx)}$$

$$= \sum_{k\in\mathbb{Z}^d} \left\| \sum_{l\in L_k} \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(I_k^r, dx)}$$

$$= \sum_{k\in\mathbb{Z}^d} \| 1 \|_{L^{p()}(I_k^r, dx)} \sum_{l\in L_k} \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)}$$

$$\overset{\text{Lemma 3.2.12 of [48]}}{\leq} \sum_{k\in\mathbb{Z}^d} \max\left\{ \left| I_k^r \right|^{\frac{1}{p_-}}, \left| I_k^r \right|^{\frac{1}{p_+}} \right\} \sum_{l\in L_k} \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)}$$

$$= \sum_{k\in\mathbb{Z}^d} \max\left\{ r^{\frac{d}{p_-}}, r^{\frac{d}{p_+}} \right\} \sum_{l\in L_k} \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)}$$

$$= \max\left\{ r^{\frac{d}{p_-}}, r^{\frac{d}{p_+}} \right\} \sum_{k\in\mathbb{Z}^d} \sum_{l\in L_k} \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)}$$

$$\overset{\text{Holder}}{\leq} K \max\left\{ r^{\frac{d}{p_-}}, r^{\frac{d}{p_+}} \right\} \sum_{k\in\mathbb{Z}^d} \left\| \{1\}_{l\in L_k} \right\|_{l^{p'()}(L_k)}$$

$$\times \left\| \left\{ \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l\in L_k} \right\|_{l^{p()}(L_k)}$$

$$\leq K \max\left\{ r^{\frac{d}{p_-}}, r^{\frac{d}{p_+}} \right\} \sum_{k\in\mathbb{Z}^d} (\text{Card } L_k)^{\frac{1}{p'_+}}$$

$$\times \left\| \left\{ \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l\in L_k} \right\|_{l^{p()}(L_k)}$$

$$= K \max\left\{ r^{\frac{d}{p_-}}, r^{\frac{d}{p_+}} \right\} (\text{Card } L_k)^{\frac{1}{p'_+}}$$

$$\times \sum_{k\in\mathbb{Z}^d} \left\| \left\{ \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l\in L_k} \right\|_{l^{p()}(L_k)}$$

$$= K \max\left\{ r^{\frac{d}{p_-}},\ r^{\frac{d}{p_+}} \right\} (\text{Card } L_k)^{\frac{1}{p'_+}}$$

$$\times \left\| \left\{ \left\| \left\{ \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l \in L_k} \right\|_{l^{p()}(L_k)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^1(\mathbb{Z}^d)}$$

$$= K \max\left\{ r^{\frac{d}{p_-}},\ r^{\frac{d}{p_+}} \right\} (\text{Card } L_k)^{\frac{1}{p'_+}}$$

$$\times \left\| \left\{ \left\| \left\{ \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \right\}_{k \in K_l} \right\|_{l^1(K_l)},$$

where for any $l \in \mathbb{Z}^d$

$K_l = \{k \in \mathbb{Z}^d : l \in L_K\} = \{k \in \mathbb{Z}^d : l_j - 1 \leq k_j \leq l_j + 1\}$, then Card $K_l \leq 3^d$, this leads to the following inequality:

$$\left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq K \max\left\{ r^{\frac{d}{p_-}},\ r^{\frac{d}{p_+}} \right\} (\text{Card } L_k)^{\frac{1}{p'_+}} \left\| \{1\}_{k \in K_l} \right\|_{l^1(K_l)}$$

$$\times \left\| \left\{ \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)},$$

that is,

$$\left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} \leq K \max\left\{ r^{\frac{d}{p_-}},\ r^{\frac{d}{p_+}} \right\} (\text{Card } L_k)^{\frac{1}{p'_+}} \text{Card } K_l$$

$$\times \left\| \left\{ \left\| f\chi_{I_l^r} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$$

or

$$\left\| \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} \leq KCC' \max\left\{ r^{\frac{d}{p_-}}, r^{\frac{d}{p_+}} \right\} {}_r\| f \|_{q(), p(), \Omega}$$

otherwise

$$\min\left\{ r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}} \right\} \cdot \left\| \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} \leq KCC' {}_r\| f \|_{q(), p(), \Omega}.$$

Thus

$$\||f\||_{q(), p(), \Omega} \leq KCC' {}_r\| f \|_{q(), p(), \Omega}, \tag{61}$$

where $C = (\text{Card } L_k)^{\frac{1}{p'_+}} \leq 3^{\frac{d}{p'_+}}$ and $C = \text{Card } K_l \leq 3^d$.

(b) Let $t = \frac{r}{2}$.

• Remark that $I_k^r = \bigcup_{l \in \{0, 1\}^d} I_{2k+l}^t$, $k \in \mathbb{Z}^d$

$$\begin{aligned} {}_r\| f \|_{q(), p(), \Omega} &= \left\| \left\{ \| f\chi_{I_k^r} \|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \\ &= \left\| \left\{ \left\| f\chi_{\bigcup_{l \in \{0, 1\}^d} I_{2k+l}^t} \right\|_{L^{q()}(\Omega)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \\ &\leq \left\| \left\{ \sum_{l \in \{0, 1\}^d} \| f\chi_{I_{2k+l}^t} \|_{L^{q()}(\Omega, dy)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \end{aligned}$$

but

$$\sum_{l \in \{0, 1\}^d} \| f\chi_{I_{2k+l}^t} \|_{L^{q()}(\Omega, dy)}$$

$$= \left\| \left\{ \left\| f\chi_{I^t_{2k+l}} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l \in \{0,1\}^d} \right\|_{l^1\{0,1\}^d}$$

$$\overset{\text{Holder ineq.}}{\leq} \left\| \{1\}_{l \in \{0,1\}^d} \right\|_{l^{p'()}\{0,1\}^d} \cdot \left\| \left\{ \left\| f\chi_{I^t_{2k+l}} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l \in \{0,1\}^d} \right\|_{l^{p()}\{0,1\}^d},$$

therefore we get

$$_r\| f \|_{q(), p(), \Omega}$$

$$\leq \left\| \{1\}_{l \in \{0,1\}^d} \right\|_{l^{p'()}\{0,1\}^d} \left\| \left\{ \left\| \left\{ \left\| f\chi_{I^t_{2k+l}} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l \in \{0,1\}^d} \right\|_{l^{p()}\{0,1\}^d} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq \left\| \{1\}_{l \in \{0,1\}^d} \right\|_{l^{p'()}\{0,1\}^d} \left\| \left\{ \left\| \left\{ \left\| f\chi_{I^t_{2k+l}} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \right\}_{k \in \{0,1\}^d} \right\|_{l^{p()}\{0,1\}^d}$$

$$= \left\| \{1\}_{l \in \{0,1\}^d} \right\|_{l^{p'()}\{0,1\}^d} \left\| \{1\}_{k \in \{0,1\}^d} \right\|_{l^{p()}\{0,1\}^d} \left\| \left\{ \left\| f\chi_{I^t_{2k+l}} \right\|_{L^{q()}(\Omega, dy)} \right\}_{l \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq \left\| \{1\}_{l \in \{0,1\}^d} \right\|_{l^{p'()}\{0,1\}^d} \left\| \{1\}_{k \in \{0,1\}^d} \right\|_{l^{p()}\{0,1\}^d} \left\| \left\{ \left\| f\chi_{I^t_{m}} \right\|_{L^{q()}(\Omega, dy)} \right\}_{m \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)},$$

therefore

$$_r\| f \|_{q(), p(), \Omega} \leq cc'\, _t\| f \|_{q(), p(), \Omega},$$

where

$$c' = \left\| \{1\}_{l \in \{0,1\}^d} \right\|_{l^{p'()}\{0,1\}^d} \leq \begin{cases} 1 \text{ if } p'_+ = \infty \\ c = \text{Card } (\{0,1\}^d)^{\frac{1}{p'_+}} \text{ if } p'_+ < \infty \end{cases} = \begin{cases} 1 \text{ if } p'_+ = \infty \\ 2^{\frac{d}{p'_+}} \text{ if } p'_+ < \infty \end{cases},$$

$$\left\| \{1\}_{l \in \{0,1\}^d} \right\|_{l^{p()}\{0,1\}^d} \leq \begin{cases} 1 \text{ if } p_+ = \infty \\ c = \text{Card } (\{0,1\}^d)^{\frac{1}{p_+}} \text{ if } p_+ < \infty \end{cases} = \begin{cases} 1 \text{ if } p_+ = \infty \\ 2^{\frac{d}{p_+}} \text{ if } p_+ < \infty \end{cases},$$

since $t = \frac{r}{2}$, we get

$$_{r}\| f \|_{q(), p(), \Omega} \leq cc' _{\frac{r}{2}}\| f \|_{q(), p(), \Omega}. \tag{62}$$

• From Lemma 5-(a)

$$x \in I_k^t \Rightarrow I_k^t \subset J_x^r, \quad x \in \Omega, \quad k \in \mathbb{Z}^d$$

and then

$$x \in I_k^t \Rightarrow \left\| f\chi_{I_k^t} \right\|_{L^{q()}(\Omega, dy)} \leq \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)}, \quad x \in \Omega, \quad k \in \mathbb{Z}^d$$

this implies that

$$\left\| \left\| f\chi_{I_k^t} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(I_k^r, dx)} \leq \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dx)} \right\|_{L^{p()}(I_k^r, dx)},$$

therefore

$$\sum_{k \in \mathbb{Z}^d} \left\| \left\| f\chi_{I_k^t} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(I_k^r, dx)} \leq \sum_{k \in \mathbb{Z}^d} \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dx)} \right\|_{L^{p()}(I_k^r, dx)},$$

then

$$\sum_{k \in \mathbb{Z}^d} \| 1 \|_{L^{p()}(I_k^r, dx)} \left\| f\chi_{I_k^t} \right\|_{L^{q()}(\Omega, dy)} \leq \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dx)} \right\|_{L^{p()}\left(\bigcup_{k \in \mathbb{Z}^d} I_k^r, dx \right)},$$

that is,

$$\sum_{k \in \mathbb{Z}^d} \| 1 \|_{L^{p()}(I_k^r, dx)} \left\| f\chi_{I_k^t} \right\|_{L^{q()}(\Omega, dy)} \leq \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dx)} \right\|_{L^{p()}(\Omega, dx)},$$

$\| 1 \|_{L^{p()}(I_k^r, dx)}$ being independent of k, we have

$$\| 1 \|_{L^{p()}(I_k^r, dx)} \sum_{k \in \mathbb{Z}^d} \left\| f\chi_{I_k^t} \right\|_{L^{q()}(\Omega, dy)} \leq \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dx)} \right\|_{L^{p()}(\Omega, dx)},$$

the last inequality remains true if $\| 1 \|_{L^{p()}(I_k^r, dx)}$ is replaced by its maximal value (see Remark 2.40 of [47]), that is,

$$\max\left\{ | I_k^r |^{\frac{1}{p_-}}, | I_k^r |^{\frac{1}{p_+}} \right\} \sum_{k \in \mathbb{Z}^d} \| f \chi_{I_k^t} \|_{L^{q()}(\Omega, dy)} \leq \left\| \| f \chi_{J_x^r} \|_{L^{q()}(\Omega, dx)} \right\|_{L^{p()}(\Omega, dx)}$$

which means that

$$\max\left\{ | I_k^r |^{\frac{1}{p_-}}, | I_k^r |^{\frac{1}{p_+}} \right\} \left\| \left\{ \| f \chi_{I_k^t} \|_{L^{q()}(\Omega, dy)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^1(\mathbb{Z}^d)} \leq \left\| \| f \chi_{J_x^r} \|_{L^{q()}(\Omega, dx)} \right\|_{L^{p()}(\Omega, dx)} \tag{63}$$

but from (8) we have

$$\left\| \left\{ \| f \chi_{I_k^t} \|_{L^{q()}(\Omega, dy)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \leq \left\| \left\{ \| f \chi_{I_k^t} \|_{L^{q()}(\Omega, dy)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^1(\mathbb{Z}^d)}. \tag{64}$$

Combining the two last numbered inequalities, we get

$$\max\left\{ | I_k^r |^{\frac{1}{p_-}}, | I_k^r |^{\frac{1}{p_+}} \right\} \left\| \left\{ \| f \chi_{I_k^t} \|_{L^{q()}(\Omega, dy)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq \left\| \| f \chi_{J_x^r} \|_{L^{q()}(\Omega, dx)} \right\|_{L^{p()}(\Omega, dx)}$$

otherwise

$$\left\| \left\{ \| f \chi_{I_k^t} \|_{L^{q()}(\Omega, dy)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq \left(\max\left\{ | I_k^r |^{\frac{1}{p_-}}, | I_k^r |^{\frac{1}{p_+}} \right\} \right)^{-1} \left\| \| f \chi_{J_x^r} \|_{L^{q()}(\Omega, dx)} \right\|_{L^{p()}(\Omega, dx)},$$

which means that

$$\left\| \left\{ \| f \chi_{I_k^t} \|_{L^{q()}(\Omega, dy)} \right\}_{k \in \mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq \min\left\{r^{\frac{-d}{p_-}},\ r^{\frac{-d}{p_+}}\right\}\left\| \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dx)} \right\|_{L^{p()}(\Omega, dx)},$$

that is,

$$_{\frac{r}{2}}\| f \|_{q(), p(), \Omega} \leq \||f\||_{q(), p(), \Omega}. \tag{65}$$

Combining (62) and (65) we will get

$$_r\| f \|_{q(), p(), \Omega} \leq cc' \||f\||_{q(), p(), \Omega}. \tag{66}$$

Combining (61) and (66) we will get

$$_r\| f \|_{q(), p(), \Omega} \leq cc' \||f\||_{q(), p(), \Omega} \leq cc'KCC'_r \| f \|_{q(), p(), \Omega}. \tag{67}$$

Case 2: $p_+ = \infty$.

In this case

$$_r\| f \|_{q(), p(), \Omega} = \||f\||_{q(), p(), \Omega} = \sup_{x\in\Omega} \| f\chi_{J_x^r} \|_{L^{q()}(\Omega)}.$$

Definition 30. Let $\mathbb{Z}^d \subset \Omega$, given $q(), \alpha() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$ such that $q() \leq \alpha() \leq p()$. We also define $\||f\||_{q(), p(), \alpha(), \Omega}$ by:

$$\||f\||_{q(), p(), \alpha(), \Omega} = \begin{cases} \sup\limits_{r>0,\, x\in\Omega} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)} - \frac{1}{p(x)}\right)} \left\| \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} & \text{if } p_+ < \infty, \\ \sup\limits_{r>0,\, x\in\Omega,\, k\in\mathbb{Z}^d} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \| f\chi_{I_k^r} \|_{L^{q()}(\Omega, dy)} & \text{if } p_+ = \infty. \end{cases} \tag{68}$$

Remark 31.

(1) Taking account of Lemma 5, we could have been defined $\||f\||_{q(), p(), \alpha(), \Omega}$ by

$$\||f\||_{q(), p(), \alpha(), \Omega} = \begin{cases} \sup\limits_{r>0,\, x\in\Omega} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)} - \frac{1}{p(x)}\right)} \left\| \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} & \text{if } p_+ < \infty, \\ \sup\limits_{r>0,\, x\in\Omega} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dy)} & \text{if } p_+ = \infty. \end{cases} \tag{69}$$

(2) In case of $p_+ = \infty$, then we can tend $p()$ to ∞, in this case in (69) we have

$$r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)} \| f\chi_{J_x^r} \|_{L^{q()}(\Omega)} \leq ||| f |||_{q(),\, p(),\, \alpha(),\, \Omega}, \quad r > 0, \quad x \in \Omega.$$

That is,

$$\| f\chi_{J_x^r} \|_{L^{q()}(\Omega)} \leq r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} ||| f |||_{q(),\, p(),\, \alpha(),\, \Omega}, \quad r > 0, \quad x \in \Omega.$$

If we tend $p()$ to ∞, we will get

$$\| f\chi_{J_x^r} \|_{L^{q()}(\Omega)} \leq^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} ||| f |||_{q(),\, \infty,\, \alpha(),\, \Omega}, \quad r > 0, \quad x \in \Omega. \tag{70}$$

The same also remains true in (68), that is,

$$\| f\chi_{I_k^r} \|_{L^{q()}(\Omega)} \leq^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} ||| f |||_{q(),\, \infty,\, \alpha(),\, \Omega}, \; r > 0, \; k \in \mathbb{Z}^d, \; x \in \Omega. \tag{71}$$

Lemma 32. [47, 48] *Given* $s() : \mathbb{R}^d \to [0, \infty)$ *such that* $s_+ < \infty$, *the following are equivalent*

(a) $s() \in LH_0(\mathbb{R}^d)$,

(b) *there exists a constant* C *depending on* d *such that given any cube* Q *and* $x \in Q$,

$$|Q|^{s(x)-s_+(Q)} \leq C \quad \text{and} \quad |Q|^{s_-(Q)-s(x)} \leq C.$$

Proposition 33. *Let* $\mathbb{Z}^d \subset \Omega$, *given* $q(), \alpha() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$ *such that* $q() \leq \alpha() \leq p()$, $p() \in LH_0(\Omega)$ (*with the zero extension of* $p()$ *on* $\Omega \setminus \mathbb{Z}^d$). *Then* $||| \cdot |||_{q(),\, p(),\, \alpha(),\, \Omega}$ *and* $\| \cdot \|_{q(),\, p(),\, \alpha(),\, \Omega}$ *are equivalent norms on* $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$.

Proof. The fact that $||| \cdot |||_{q(),\, p(),\, \alpha(),\, \Omega}$ is a norm, follows from its definition.

By the precedent Proposition 29, there exist two positive real numbers A and B such that

$$A_r \| f \|_{q(), p(), \Omega} \leq \||\, f \,\||_{q(), p(), \Omega} \leq B_r \| f \|_{q(), p(), \Omega}, \quad f \in L^{q()}_{loc}.$$

For any $x \in \Omega$, multiplying this double inequality by $r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)}$ we will get:

$$Ar^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)}{}_r \| f \|_{q(), p(), \Omega} \leq r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \||\, f \,\||_{q(), p(), \Omega}$$

$$\leq Br^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)}{}_r \| f \|_{q(), p(), \Omega},$$

if we pass to the supremum over all $r > 0$, $x \in \Omega$, we will get:

$$A \| f \|_{q(), p(), \alpha(), \Omega} \leq \sup_{r > 0,\, x \in \Omega} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \||\, f \,\||_{q(), p(), \Omega} \leq B \| f \|_{q(), p(), \alpha(), \Omega}.$$

It remains to prove that

$$\||\, f \,\||_{q(), p(), \alpha(), \Omega} \equiv \sup_{r > 0,\, x \in \Omega} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \||\, f \,\||_{q(), p(), \Omega},$$

that is, there exist positive real numbers c, C such that

$$c \sup_{r > 0,\, x \in \Omega} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \||\, f \,\||_{q(), p(), \Omega} \leq \||\, f \,\||_{q(), p(), \alpha(), \Omega}$$

$$\leq C \sup_{r > 0,\, x \in \Omega} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \||\, f \,\||_{q(), p(), \Omega},$$

or

$$\||\, f \,\||_{q(), p(), \alpha(), \Omega} \equiv \sup_{r > 0,\, x \in \Omega} \left\{ r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \min\left(r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}} \right) \right\}$$

$$\times \left\| \left\| f \chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)},$$

otherwise

$$\sup_{r>0,\, x\in\Omega} r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p(x)}\right)} \left\| \| f\chi_{J_x^r} \|_{L^{q(\cdot)}(\Omega, dy)} \right\|_{L^{p(\cdot)}(\Omega, dx)}$$

$$\equiv \sup_{r>0,\, x\in\Omega} \left\{ r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)} \cdot \min\left(r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}} \right) \right\} \left\| \| f\chi_{J_\cdot^r} \|_{L^{q(\cdot)}(\Omega)} \right\|_{L^{p(\cdot)}(\Omega)}.$$

We have for any $x \in \Omega$:

$$\frac{d}{p_+} \leq \frac{d}{p(x)} \leq \frac{d}{p_-} \quad \text{or} \quad -\frac{d}{p_-} \leq -\frac{d}{p(x)} \leq -\frac{d}{p_+}. \tag{72}$$

Sub-case 1: $0 < r < 1$.

In this sub-case

$$r^{-\frac{d}{p_+}} \leq r^{-\frac{d}{p(x)}} \leq r^{-\frac{d}{p_-}} \quad \text{and} \quad \min\left(r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}} \right) = r^{-\frac{d}{p_+}},$$

and we can write

$$r^{-\frac{d}{p_+}} \leq \min\left(r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}} \right) \leq r^{-\frac{d}{p_-}},$$

from this we get

$$r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p_+}\right)} \leq \min\left(r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}} \right) \cdot r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)} \leq r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p_-}\right)}.$$

Multiplying this double inequality by $\left\| \| f\chi_{J_x^r} \|_{L^{q(\cdot)}(\Omega, dy)} \right\|_{L^{p(\cdot)}(\Omega, dx)}$, we will get:

$$r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p_+}\right)} \left\| \| f\chi_{J_x^r} \|_{L^{q(\cdot)}(\Omega, dy)} \right\|_{L^{p(\cdot)}(\Omega, dx)}$$

$$\leq \min\left\{r^{\frac{-d}{p_-}},\ r^{\frac{-d}{p_+}}\right\}\cdot r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)}\left\|\left\| f\chi_{J_x^r}\right\|_{L^{q()}(\Omega, dy)}\right\|_{L^{p()}(\Omega, dx)}$$

$$\leq r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p_-}\right)}\left\|\left\| f\chi_{J_x^r}\right\|_{L^{q()}(\Omega, dy)}\right\|_{L^{p()}(\Omega, dx)}.$$

This last double inequality is equivalent to

$$r^{d\left(\frac{1}{p(x)}-\frac{1}{p_+}\right)} r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p(x)}\right)}\left\|\left\| f\chi_{J_x^r}\right\|_{L^{q()}(\Omega, dy)}\right\|_{L^{p()}(\Omega, dx)}$$

$$\leq \min\left\{r^{\frac{-d}{p_-}},\ r^{\frac{-d}{p_+}}\right\} r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)}\left\|\left\| f\chi_{J_x^r}\right\|_{L^{q()}(\Omega, dy)}\right\|_{L^{p()}(\Omega, dx)}$$

$$\leq r^{d\left(\frac{1}{p(x)}-\frac{1}{p_-}\right)} r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p(x)}\right)}\left\|\left\| f\chi_{J_x^r}\right\|_{L^{q()}(\Omega, dy)}\right\|_{L^{p()}(\Omega, dx)},$$

in other hand since $x \in J_x^r,\ p() \in LH_0(\Omega),\ p_+ < \infty,$ we have:

$$\begin{cases} r^{d\left(\frac{1}{p(x)}-\frac{1}{p_+}\right)} = \left|J_x^r\right|^{\left(\frac{1}{p(x)}-\frac{1}{p_+}\right)} = \left|J_x^r\right|^{\frac{p_+-p(x)}{p(x)p_+}} = \left(\left|J_x^r\right|^{p(x)-p_+}\right)^{\frac{-1}{p(x)p_+}} = C^{\frac{-1}{p(x)\cdot p_+}} \leq \max\left\{C^{\frac{-1}{p_-\times p_+}},\ C^{\frac{-1}{p_+\times p_+}}\right\} \\ r^{d\left(\frac{1}{p(x)}-\frac{1}{p_-}\right)} = \left|J_x^r\right|^{\left(\frac{1}{p(x)}-\frac{1}{p_-}\right)} = \left|J_x^r\right|^{\frac{p_--p(x)}{p(x)p_-}} = \left(\left|J_x^r\right|^{p_--p(x)}\right)^{\frac{1}{p(x)p_-}} = C^{\frac{-1}{p(x)\cdot p_-}} \leq \max\left\{C^{\frac{1}{p_-\times p_+}},\ C^{\frac{1}{p_-\times p_-}}\right\} \end{cases},$$

that is,

$$\begin{cases} r^{d\left(\frac{1}{p(x)}-\frac{1}{p_+}\right)} \leq \max\left\{C^{\frac{-1}{p_-\times p_+}},\ C^{\frac{-1}{(p_+)^2}}\right\} = K_1 < \infty \\ r^{d\left(\frac{1}{p(x)}-\frac{1}{p_-}\right)} \leq \max\left\{C^{\frac{1}{p_-\times p_+}},\ C^{\frac{1}{(p_-)^2}}\right\} = K_2 < \infty \end{cases}.$$

This leads to

$$K_1 r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p(x)}\right)}\left\|\left\| f\chi_{J_x^r}\right\|_{L^{q()}(\Omega, dy)}\right\|_{L^{p()}(\Omega, dx)}$$

$$\leq \min\left\{r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}}\right\} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \left\| \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq K_2 r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)} - \frac{1}{p(x)}\right)} \left\| \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}.$$

This means that

$$K_1 \| f \|_{q(), p(), \alpha(), \Omega} \leq \| f \|_{q(), p(), \alpha(), \Omega} \leq K_2 \| f \|_{q(), p(), \alpha(), \Omega}.$$

Sub-case 2: $r \geq 1$:

In this case, using (72), we get

$$r^{-\frac{d}{p_-}} \leq r^{-\frac{d}{p(x)}} \leq r^{-\frac{d}{p_+}} \quad \text{and} \quad \min\left\{r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}}\right\} = r^{-\frac{d}{p_-}},$$

and we can write

$$r^{-\frac{d}{p_-}} \leq \min\left\{r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}}\right\} \leq r^{-\frac{d}{p_+}}.$$

From this we get

$$r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)} - \frac{1}{p_-}\right)} \leq \min\left\{r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}}\right\} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \leq r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)} - \frac{1}{p_+}\right)}.$$

Multiplying this double inequality by $\left\| \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$ we will get:

$$r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)} - \frac{1}{p_-}\right)} \left\| \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq \min\left\{r^{\frac{-d}{p_-}}, r^{\frac{-d}{p_+}}\right\} r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)}\right)} \left\| \| f\chi_{J_x^r} \|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p+}\right)} \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}.$$

This last double inequality is equivalent to

$$r^{d\left(\frac{1}{p(x)}-\frac{1}{p_-}\right)} r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p(x)}\right)} \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq \min\left\{ r^{\frac{-d}{p_-}},\ r^{\frac{-d}{p_+}} \right\} r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)} \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq r^{d\left(\frac{1}{p(x)}-\frac{1}{p_+}\right)} r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p(x)}\right)} \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}.$$

In other hand since $x \in J_x^r$, $p() \in LH_0(\Omega)$, $p_+ < \infty$, we have:

$$\begin{cases} r^{d\left(\frac{1}{p(x)}-\frac{1}{p_+}\right)} = |J_x^r|^{\left(\frac{1}{p(x)}-\frac{1}{p_+}\right)} = |J_x^r|^{\frac{p_+-p(x)}{p(x)p_+}} = \left(|J_x^r|^{p(x)-p_+}\right)^{\frac{-1}{p(x)p_+}} \overset{\text{Lem. } 32}{=} C^{\frac{-1}{p(x)\cdot p_+}} \leq \max\left\{ C^{\frac{-1}{p_-p_+}},\ C^{\frac{-1}{p_+p_+}} \right\} \\ r^{d\left(\frac{1}{p(x)}-\frac{1}{p_-}\right)} = |J_x^r|^{\left(\frac{1}{p(x)}-\frac{1}{p_-}\right)} = |J_x^r|^{\frac{p_--p(x)}{p(x)p_-}} = \left(|J_x^r|^{p_--p(x)}\right)^{\frac{1}{p(x)p_-}} \overset{\text{Lem. } 32}{=} C^{\frac{-1}{p(x)\cdot p_-}} \leq \max\left\{ C^{\frac{1}{p_-p_+}},\ C^{\frac{1}{p_-p_-}} \right\} \end{cases},$$

that is,

$$\begin{cases} r^{d\left(\frac{1}{p(x)}-\frac{1}{p_+}\right)} \leq \max\left\{ C^{\frac{-1}{p_-p_+}},\ C^{\frac{-1}{(p_+)^2}} \right\} = K_1 < \infty \\ r^{d\left(\frac{1}{p(x)}-\frac{1}{p_-}\right)} \leq \max\left\{ C^{\frac{1}{p_-p_+}},\ C^{\frac{1}{(p_-)^2}} \right\} = K_2 < \infty \end{cases}.$$

This leads to

$$K_2 r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}-\frac{1}{p(x)}\right)} \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq \min\left\{ r^{\frac{-d}{p_-}},\ r^{\frac{-d}{p_+}} \right\} r^{d\left(\frac{1}{\alpha(x)}-\frac{1}{q(x)}\right)} \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq K_1 r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)} - \frac{1}{p(x)}\right)} \left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}.$$

This means that

$$K_2 \| f \|_{q(), p(), \alpha(), \Omega} \leq \||| f \|||_{q(), p(), \alpha(), \Omega} \leq K_1 \| f \|_{q(), p(), \alpha(), \Omega}$$

and the claim is proved.

4. Riesz Potential on $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$

We begin by recalling some classical results, first one definition is given with a variant of definition of the Riesz potential.

Definition 34. [47] Given γ, $0 < \gamma < 1$, define the Riesz potential I_γ, also referred to as the fractional integral operator with index γ, to be the convolution operator

$$I_\gamma f(x) = \int_{\mathbb{R}^d} | x - y |^{d(\gamma - 1)} f(y) dy.$$

The Riesz potentials are not bounded on $L^p(\mathbb{R}^d)$, but satisfy off-diagonal inequalities.

With this definition, we have the following classical results:

Theorem 35. [47] *Given* γ, $0 < \gamma < 1$ *and* $1 \leq p < \frac{d}{\gamma}$, *define* $q > p$ *by* $\frac{1}{p} - \frac{1}{q} = \gamma$.

If $p = 1$, *then for all* $t > 0$

$$\left| \{x \in \mathbb{R}^d : \left| I_\gamma f(x) \right| > t\} \right| \leq \left(\frac{C}{t} \int_{\mathbb{R}^d} f(x) dx \right)^q.$$

If $p > 1$, *then*

$$\left\| I_\gamma f \right\|_q \leq C \| f \|_p.$$

The Riesz potentials are well defined on the variable Lebesgue spaces. If $p_+ < \frac{d}{\gamma}$ and $f \in L^{p()}(\mathbb{R}^d)$, then $I_\gamma f(x)$ converges for every x. Theorem 35 can be extended to the variable Lebesgue spaces in the following manner.

Theorem 36. [47] *Fix* γ, $0 < \gamma < 1$. *Given* $p() \in \mathcal{P}(\mathbb{R}^d)$ *such that* $1 < p_- \leq p_+ < \frac{1}{\gamma}$, *define* $q()$ *by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \gamma.$$

If there exists $q_0 > \frac{1}{1-\gamma}$ *such that the Hardy-Littlewood maximal operator* M *is bounded on* $L^{(q()/q_0)'}(\mathbb{R}^d)$, *then*

$$\| I_\gamma f \|_{q()} \leq C \| f \|_{p()}.$$

If $p_- = 1$ *and* M *is bounded on* $L^{(q()/q_0)'}(\mathbb{R}^d)$ *when* $q_0 = \frac{1}{1-\gamma}$, *then for every* $t > 0$,

$$\left\| t\chi_{\{x \in \mathbb{R}^d : |I_\gamma f(x)| > t\}} \right\|_{q()} \leq C \| f \|_{p()}.$$

We will need the following lemma:

Lemma 37. (Minkowski's integral inequality for variable Lebesgue spaces [47])

Let $q() \in \mathcal{P}(\Omega)$, *let* $f : \Omega \times \Omega \to \mathbb{R}$ *be a measurable function* (*with respect to product measure*) *such that for almost every* $y \in \Omega$, $f(., y) \in L^{q()}(\Omega)$. *Then*

$$\left\| \int_\Omega f(., y) dy \right\|_{L^{q()}(\Omega, dx)} \leq C \int_\Omega \| f(., y) \|_{L^{q()}(\Omega, dx)} dy. \tag{73}$$

Proposition 38. *Let* $\mathbb{Z}^d \subset \Omega$, $q(), \alpha() \in \mathcal{P}(\Omega)$, $q() \in \mathcal{P}(\Omega) \cap$

$LH_0(\Omega)$, and $1 < q_- \leq q() \leq \alpha() \leq p() \leq \infty$, $\left(\frac{1}{q} + \frac{1}{p} - \frac{1}{\alpha}\right)_+ < \infty$. Suppose that there exists $q_0 > \frac{1}{1-\gamma}$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{\left(\frac{q()}{q_0}\right)'}(\Omega)$ and

$$\begin{cases} 0 < \gamma < \frac{1}{\alpha()} - \frac{1}{p()}, \\ 0 < \frac{1}{q^*()} = \frac{1}{q()} - \gamma, \\ 0 < \frac{1}{\alpha^*()} = \frac{1}{\alpha} - \gamma. \end{cases}$$

Then for any $f \in (L^{q()}, l^{p()})^{\alpha()}(\Omega)$, $I_\gamma f$ belongs to $(L^{q^*()}, l^{p()})^{\alpha^*()}(\Omega)$ and

$$\| I_\gamma f \|_{q^*(), p(), \alpha^*(), \Omega} \leq C \| f \|_{q(), p(), \alpha(), \Omega}, \tag{74}$$

where C is a constant not depending on f.

Proof. The proof of Proposition 38 is an adaptation of that of Proposition 4.1 in [15].

For any real numbers $a, b, c,$ we have:

$$| c - b | \geq | | c - a | - | a - b | |. \tag{β_1}$$

(a) Case $p_+ = \infty$.

When $p_+ = \infty$, $p()$ is unbounded, let f be a positive element of $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$, we have

$$\| f \|_{q(), \infty, \alpha(), \Omega} \overset{\text{Propo. 23-(2)}}{\leq} C \| f \|_{q(), p(), \alpha(), \Omega} < \infty.$$

Let $(x, r) \in \Omega \times (0, \infty)$. We have:

$$f = \sum_{n \in \mathbb{N}_0} f_{x, r, n},$$

where $f_{x,r,0} = f\chi_{J_x^{2r}}$ and $f_{x,r,n} = f\chi_{T_{x,r,n}}$ where $T_{x,r,n} = J_x^{2^{n+1}r} \setminus J_x^{2^n r}$ for $n \in \mathbb{N}$.

Since f is positive, monotone convergence theorem can be applied to get

$$I_\gamma f = \sum_{n \in \mathbb{N}_0} I_\gamma f_{x,r,n}.$$

First:

From Hardy-Littlewood-Sobolev theorem for fractional integral Theorem 36, there exists a positive constant A not depending on f and r such that

$$\| I_\gamma f_{x,r,0} \|_{L^{q^*()}(\Omega)} \leq A \| f_{x,r,0} \|_{L^{q()}(\Omega)} = A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega)}.$$

Secondly:

$$\begin{cases} \forall n \in \mathbb{N}: \\ y \in T_{x,r,n} = J_x^{2^{n+1}r} \setminus J_x^{2^n r} \\ z \in J_x^r \end{cases} \Rightarrow \begin{cases} \sqrt{d}\, 2^{n-1} r \leq | x - y | \leq \sqrt{d}\, 2^n r \\ 0 \leq | x - z | \leq \sqrt{d}\, 2^{-1} r \end{cases}$$

$$\overset{(\beta_1)}{\Rightarrow} \quad | z - y | \geq \big| | x - y | - | x - z | \big|,$$

this implies that

$$| z - y | \geq \sqrt{d}\, 2^{n-1} r - \sqrt{d}\, 2^{-1} r \geq \frac{2^n r}{2} - \frac{r}{2} = (2^n - 1)\frac{r}{2},$$

that is,

$$| z - y | \geq 2^{n-1} r. \tag{β_2}$$

To little simplify the notation, we write $\| f \|_{q(), p(), \alpha(), \Omega} = \| f \|_{q(), p(), \alpha()}$, the following notations should be understood in the sense:

$$q_-^* = (q^*())_- = (q^*)_-, \; q_+^* = (q^*())_+ = (q^*)_+,$$

$$q'_- = (q'())_- = (q')_-, \; q'_+ = (q'())_+ = (q')_+. \tag{75}$$

Recall that in Section 2 (Definitions and Notations), it is clearly written that

$$(p'())_+ = (p_-)', \quad (p'())_- = (p_+)'. \tag{76}$$

In the following proof we suppose that the one-dot $(\cdot)$ and two-dot $(\cdot\,\cdot)$, respectively, stand for the variables z and y, therefore

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)}$$

$$\leq \sum_{n\geq 0} \left\| (I_\gamma f_{x,r,n})\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)}$$

$$\leq A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega)}$$

$$+ \sum_{n\geq 1} \left\| \left\| |\cdot - \cdot\,\cdot|^{d(\gamma-1)} f(\cdot\,\cdot) \right\|_{L^1(T_{x,r,n},\,dy)} \right\|_{L^{q^*()}(J_x^r,\,dz)}$$

$$\overset{\text{Minkowski}}{\leq} \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega)}$$

$$+ \sum_{n\geq 1} \left\| \left\| |\cdot - \cdot\,\cdot|^{d(\gamma-1)} f(\cdot\,\cdot) \right\|_{L^{q^*()}(J_x^r,\,dz)} \right\|_{L^1(T_{x,r,n},\,dy)}$$

$$\overset{(\beta_2)}{\leq} A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega)} \sum_{n\geq 1} (2^{n-1}r)^{d(\gamma-1)} \| 1 \|_{L^{q^*()}(J_x^r,\,dz)}$$

$$\times \left\| f(\cdot\,\cdot)\chi_{T_{x,r,n}} \right\|_{L^1(\Omega,\,dy)}$$

$$\overset{\text{Holder}}{\leq} A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega)} + \sum_{n\geq 1} (2^{n-1}r)^{d(\gamma-1)} \| 1 \|_{L^{q^*()}(J_x^r,\,dz)}$$

$$\times \| 1 \|_{L^{q'()}(T_{x,r,n})} \| f(\cdot\,\cdot) \|_{L^{q()}(T_{x,r,n},\,dy)}$$

$$\overset{\text{Lem. 3.2.12 of [48]}}{\leq} A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega)}$$

$$+\sum_{n\geq 1}(2^{n-1}r)^{d(\gamma-1)}\max\left\{|J_x^r|^{\frac{1}{q_-^*}}, |J_x^r|^{\frac{1}{q_+^*}}\right\}$$

$$\times\max\left\{|T_{x,r,n}|^{\frac{1}{q'_-}}, |T_{x,r,n}|^{\frac{1}{q'_+}}\right\}\|f\|_{L^{q()}(T_{x,r,n},dy)}$$

$$= A\left\|f\chi_{J_x^{2r}}\right\|_{L^{q()}(\Omega)}+\sum_{n\geq 1}(2^{n-1}r)^{d(\gamma-1)}\max\left\{r^{\frac{d}{q_-^*}}, r^{\frac{d}{q_+^*}}\right\}$$

$$\times\max\left\{((2^{n+1}r)^d-(2^nr)^d)^{\frac{1}{q'_-}}, ((2^{n+1}r)^d-(2^nr)^d)^{\frac{1}{q'_+}}\right\}$$

$$\times\|f\|_{L^{q()}(J_x^{2^{n+1}r},dy)}$$

$$\leq A\left\|f\chi_{J_x^{2r}}\right\|_{L^{q()}(\Omega)}+\sum_{n\geq 1}(2^{n-1}r)^{d(\gamma-1)}\max\left\{r^{\frac{d}{q_-^*}}, r^{\frac{d}{q_+^*}}\right\}$$

$$\times\max\left\{\left[(2^nr)^d(2^d-1)\right]^{\frac{1}{q'_-}}, \left[(2^nr)^d(2^d-1)\right]^{\frac{1}{q'_+}}\right\}$$

$$\times\|f\|_{L^{q()}(J_x^{2^{n+1}r},dy)}$$

$$\leq A\left\|f\chi_{J_x^{2r}}\right\|_{L^{q()}(\Omega)}+\sum_{n\geq 1}(2^{n-1}r)^{d(\gamma-1)}\max\left\{r^{\frac{d}{q_-^*}}, r^{\frac{d}{q_+^*}}\right\}$$

$$\times\max\{(2^nr)^{d/q'_-}(2^{d/q'_-}), (2^nr)^{d/q'_+}(2^{d/q'_+})\}\|f\|_{L^{q()}(J_x^{2^{n+1}r},dy)}.$$

That is,

$$\left\|(I_\gamma f)\chi_{J_x^r}\right\|_{L^{q^*()}(\Omega)}\leq A\left\|f\chi_{J_x^{2r}}\right\|_{L^{q()}(\Omega)}+\sum_{n\geq 1}(2^{n-1}r)^{d(\gamma-1)}\max\left\{r^{\frac{d}{q_-^*}}, r^{\frac{d}{q_+^*}}\right\}$$

$$\times\max\left\{(2^{n+1})^{\frac{d}{q'_-}}r^{\frac{d}{q'_-}}, (2^{n+1})^{\frac{d}{q'_+}}r^{\frac{d}{q'_+}}\right\}\|f\|_{L^{q()}(J_x^{2^{n+1}r},dy)}. \tag{77}$$

From here we will envisage two cases:

First case: $0 < r < 1.$

In this case $\max\left\{ r^{\frac{d}{q_-^*}}, r^{\frac{d}{q_+^*}} \right\} = r^{\frac{d}{q_+^*}}$ and

$$\max\left\{ (2^{n+1})^{\frac{d}{q_-'}} r^{\frac{d}{q_-'}}, (2^{n+1})^{\frac{d}{q_+'}} r^{\frac{d}{q_+'}} \right\} \leq (2^{n+1})^{\frac{d}{q_-'}} r^{\frac{d}{q_+'}}.$$

Then the last numbered inequality is bounded by

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega)}$$

$$+ \sum_{n \geq 1} (2^{n-1} r)^{d(\gamma - 1)} r^{\frac{d}{q_+^*}} \times (2^{n+1})^{\frac{d}{q_-'}} r^{\frac{d}{q_+'}} \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)}$$

$$= A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega)} + 2^{d\left(1 - \gamma + \frac{1}{q_-'}\right)} r^{d\left(\gamma - 1 + \frac{1}{q_+^*} + \frac{1}{q_+'}\right)} \sum_{n \geq 1} 2^{nd\left(\gamma - 1 + \frac{1}{q_-'}\right)}$$

$$\times \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)}.$$

That is,

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega)}$$

$$+ 2^{-d\left(\gamma + \frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_+} - \frac{1}{q_-}\right)} \sum_{n \geq 1} 2^{-\frac{nd}{q_-^*}} \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)}. \qquad (\beta_3)$$

Using (24) we will get

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)}$$

$$\leq A(2r)^{d\left(\frac{1}{q(x)} - \frac{1}{\alpha(x)}\right)} \| f \|_{q(), \infty, \alpha()}$$

$$+\, 2^{-d\left(\gamma+\frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_+}-\frac{1}{q_-}\right)} \sum_{n\geq 1} 2^{-\frac{nd}{q_-^*}} (2^{n+1} r)^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),\infty,\alpha()}$$

$$= 2^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \left[A + 2^{-d\left(\gamma+\frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_+}-\frac{1}{q_-}\right)} \sum_{n\geq 1} 2^{nd\left(\frac{1}{q(x)}-\frac{1}{q_-^*}-\frac{1}{\alpha(x)}\right)} \right]$$

$$\times\, r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),\infty,\alpha()}$$

$$= 2^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \left[A + 2^{-d\left(\gamma+\frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_+}-\frac{1}{q_-}\right)} \sum_{n\geq 1} 2^{nd\left(\gamma-\frac{1}{\alpha(x)}+\frac{1}{q(x)}-\frac{1}{q_-}\right)} \right]$$

$$\times\, r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),\infty,\alpha()}$$

$$\leq 2^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \left[A + 2^{-d\left(\gamma+\frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_+}-\frac{1}{q_-}\right)} \sum_{n\geq 1} 2^{nd\left(-\frac{1}{\alpha^*(x)}+\frac{1}{q(x)}-\frac{1}{q_-}\right)} \right]$$

$$\times\, r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),\infty,\alpha()}.$$

That is,

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq 2^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} \left[A + 2^{-d\left(\gamma+\frac{1}{q_+}\right)} \beta(r) \sum_{n\geq 1} 2^{nd\eta(x)} \right]$$

$$\times\, r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(),\infty,\alpha()} \tag{78}$$

with

$$\beta(r) = r^{d\left(\frac{1}{q_+}-\frac{1}{q_-}\right)}, \quad \eta(x) = -\frac{1}{\alpha^*(x)} + \frac{1}{q(x)} - \frac{1}{q_-}.$$

It is easy to see that $\eta(x) \leq 0$ and $\beta(r) = r^{d\left(\frac{1}{q_+} - \frac{1}{q_-}\right)}$ is independent of r, in fact, we prove it:

For any $x \in \Omega$, we have $q_- \leq q(x)$, then $\frac{1}{q_-} \geq \frac{1}{q(x)}$ or $\frac{1}{q(x)} - \frac{1}{q_-} \leq 0$, since $-\frac{1}{\alpha^*(x)} < 0$, we get $\eta(x) \leq 0$, this implies that $\sum_{n \geq 1} 2^{nd\eta(x)}$ is a convergent series, we let $\sum_{n \geq 1} 2^{nd\eta(x)} = K < \infty$. $\beta(r) = r^{d\left(\frac{1}{q_+} - \frac{1}{q_-}\right)}$ $= \left| J_x^r \right|^{\frac{1}{q_+} - \frac{1}{q_-}} = \left| J_x^r \right|^{\frac{1}{q_+} - \frac{1}{q(x)}} \times \left| J_x^r \right|^{\frac{1}{q(x)} - \frac{1}{q_-}}$, by hypotheses we have $1 < q_- \leq q() \leq \alpha() \leq p() \leq \infty$, this implies that $q_+ \leq \alpha() < p() \leq \infty$, that is, $q_+ < \infty$, therefore the hypotheses of Lemma 3.24 of [48] are satisfied, then

$$\begin{aligned}
\beta(r) &= \left(\left| J_x^r \right|^{q(x) - q_+} \right)^{\frac{1}{q(x) q_+}} \times \left(\left| J_x^r \right|^{q_- - q(x)} \right)^{\frac{1}{q(x) q_-}} \\
&\overset{\text{Lem. 3.24 of [48]}}{\leq} (C_1(d))^{\frac{1}{q(x) q_+}} \times (C_2(d))^{\frac{1}{q(x) q_-}} \\
&\leq \max\left\{ (C_1(d))^{\frac{1}{q_- q_+}} \times (C_2(d))^{\frac{1}{q_- q_-}},\ (C_1(d))^{\frac{1}{q_+ q_+}} \times (C_2(d))^{\frac{1}{q_+ q_-}} \right\} \\
&= B < \infty,
\end{aligned}$$

then the last numbered inequality becomes

$$\left\| (I_\gamma f) \chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq 2^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_+} \left[A + 2^{-d\left(\gamma + \frac{1}{q_+}\right)} BK \right] r^{d\left(\frac{1}{q(x)} - \frac{1}{\alpha(x)}\right)} \| f \|_{q(), \infty, \alpha()}, \tag{79}$$

where $2^{d\left(\frac{1}{q} - \frac{1}{\alpha}\right)_+} = 2^{d\left(\frac{1}{q} + \frac{1}{\infty} - \frac{1}{\alpha}\right)_+} = 2^{d\left(\frac{1}{q} + \frac{1}{p} - \frac{1}{\alpha}\right)_+} < \infty$ since

$$\left(\frac{1}{q} + \frac{1}{p} - \frac{1}{\alpha} \right)_+ < \infty, \ 2^{-d\left(\gamma + \frac{1}{q_+}\right)} < \infty.$$

If we let $C(q(), \infty, \alpha) = 2^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}\left[A + 2^{-d\left(\gamma+\frac{1}{q_+}\right)}BK\right]$, we have $C(q(), \infty, \alpha) < \infty$ and the last numbered inequality becomes

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq C(q(), \infty, \alpha) r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(), \infty, \alpha()}.$$

By hypotheses we have $\frac{1}{\alpha(x)} - \frac{1}{q(x)} = \frac{1}{\alpha^*(x)} - \frac{1}{q^*(x)}$, therefore

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq C(q(), \infty, \alpha) r^{d\left(\frac{1}{q^*(x)}-\frac{1}{\alpha^*(x)}\right)} \| f \|_{q(), \infty, p(), \Omega},$$

this implies that

$$r^{d\left(\frac{1}{\alpha^*(x)}-\frac{1}{q^*(x)}\right)} \left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq C(q(), \infty, \alpha) \| f \|_{q(), \infty, p(), \Omega}.$$

Second case: $1 \leq r < +\infty$.

In this case $\max\left\{ r^{\frac{d}{q_-^*}}, r^{\frac{d}{q_+^*}} \right\} = r^{\frac{d}{q_-^*}}$ and

$$\max\left\{ (2^{n+1})^{\frac{d}{q'_-}} r^{\frac{d}{q'_-}}, (2^{n+1})^{\frac{d}{q'_+}} r^{\frac{d}{q'_+}} \right\} = (2^{n+1})^{\frac{d}{q'_-}} r^{\frac{d}{q'_-}}.$$

Then, taking account of (75) and (76) the right-hand side of (77) is bounded by:

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega)} + \sum_{n \geq 1} (2^{n-1} r)^{d(\gamma-1)} r^{\frac{d}{q_-^*}}$$

$$\times (2^{n+1})^{\frac{d}{q'_-}} r^{\frac{d}{q_-^*}} \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)},$$

that is,

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*(\cdot)}(\Omega)} \leq A\left\| f\chi_{J_x^{2r}} \right\|_{L^{q(\cdot)}(\Omega)} + 2^{d\left(2-\gamma-\frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_-}-\frac{1}{q_+}\right)}$$

$$\times \sum_{n\geq 1} 2^{-\frac{nd}{q_-^*}} \| f \|_{L^{q(\cdot)}(J_x^{2^{n+1}r}, dy)}. \qquad (\beta_4)$$

Using (24) we will get

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*(\cdot)}(\Omega)} \leq$$

$$2^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)}\left[A + 2^{d\left(2-\gamma-\frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_-}-\frac{1}{q_+}\right)} \sum_{n\geq 1} 2^{nd\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}-\frac{1}{q_-^*}\right)}\right] r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(\cdot),\infty,\alpha(\cdot)}$$

$$= 2^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)}\left[A + 2^{d\left(2-\gamma-\frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_-}-\frac{1}{q_+}\right)} \sum_{n\geq 1} 2^{nd\left(\gamma-\frac{1}{\alpha(x)}+\frac{1}{q(x)}-\frac{1}{q_-}\right)}\right] r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(\cdot),\infty,\alpha(\cdot)}$$

$$= 2^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)}\left[A + 2^{d\left(2-\gamma-\frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_-}-\frac{1}{q_+}\right)} \sum_{n\geq 1} 2^{nd\left(\frac{-1}{\alpha^*(x)}+\frac{1}{q(x)}-\frac{1}{q_-}\right)}\right] r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(\cdot),\infty,\alpha(\cdot)},$$

that is,

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*(\cdot)}(\Omega)} \leq 2^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}\left[A + 2^{d\left(2-\gamma-\frac{1}{q_+}\right)} B_0(r) \sum_{n\geq 1} 2^{nd\eta(x)}\right]$$

$$\times r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(\cdot),\infty,\alpha(\cdot)} \qquad (80)$$

with

$$B_0(r) = r^{d\left(\frac{1}{q_-}-\frac{1}{q_+}\right)}, \quad \eta(x) = \frac{-1}{\alpha^*(x)} + \frac{1}{q(x)} - \frac{1}{q_-}.$$

We can remark that $B_0(r) = (B(r))^{-1}$, we already have proved in the first case that $B(r)$ is bounded by a constant which does not depend on r, so is $B_0(r)$, therefore $B_0(r) = B_0$, for any $x \in \Omega$ we have $q_- \leq q(x)$, then $\frac{1}{q_-} \geq \frac{1}{q(x)}$ or $\frac{1}{q(x)} - \frac{1}{q_-} \leq 0$, since $-\frac{1}{\alpha^*(x)} < 0$ we get $\eta(x) \leq 0$, this implies that $\sum_{n\geq 1} 2^{nd\eta(x)}$ is a convergent series, we let $\sum_{n\geq 1} 2^{nd\eta(x)} = K < \infty$, therefore the last numbered inequality becomes

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq 2^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}\left[A + 2^{d\left(2-\gamma-\frac{1}{q_+}\right)}B_0K\right] r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)}\| f \|_{q(), \infty, \alpha()},$$

where

$$2^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+} = 2^{d\left(\frac{1}{q}+\frac{1}{\infty}-\frac{1}{\alpha}\right)_+} = 2^{d\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\alpha}\right)_+} < \infty$$

since $\left(\frac{1}{q} + \frac{1}{p} - \frac{1}{\alpha}\right)_+ < \infty$, $2^{d\left(2-\gamma-\frac{1}{q_+}\right)} < \infty$.

If we let $C(q(), \infty, \alpha()) = 2^{d\left(\frac{1}{q}-\frac{1}{\alpha}\right)_+}\left[A + 2^{d\left(2-\gamma-\frac{1}{q_+}\right)}B_0K\right]$, we have $C(q(), \infty, \alpha()) < \infty$ and the last numbered inequality becomes

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq C(q(), \infty, \alpha) r^{d\left(\frac{1}{q(x)}-\frac{1}{\alpha(x)}\right)}\| f \|_{q(), \infty, \alpha()}.$$

By hypotheses we have $\frac{1}{\alpha(x)} - \frac{1}{q(x)} = \frac{1}{\alpha^*(x)} - \frac{1}{q^*(x)}$, therefore

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq C(q(), \infty, \alpha()) r^{d\left(\frac{1}{q^*(x)}-\frac{1}{\alpha^*(x)}\right)}\| f \|_{q(), \infty, \alpha(), \Omega},$$

this implies that

$$r^{d\left(\frac{1}{\alpha^*(x)}-\frac{1}{q^*(x)}\right)}\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq C(q(), \infty, \alpha())\| f \|_{q(), \infty, \alpha(), \Omega}.$$

In both cases $0 < r < 1$ and $1 \leq r < \infty$, we have

$$r^{d\left(\frac{1}{\alpha^*(x)} - \frac{1}{q^*(x)}\right)} \left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega)} \leq C(q(), \infty, \alpha) \| f \|_{q(), \infty, \alpha(), \Omega}.$$

Now taking account of (23), if we pass to the supremum over all $r > 0$, $x \in \Omega$, we get

$$\| I_\gamma f \|_{q^*(), \infty, \alpha^*(), \Omega} \leq C(q(), \infty, \alpha) \| f \|_{q(), \infty, \alpha(), \Omega}.$$

(b) Case $p_+ < \infty$.

Consider $f \in (L^{q()}, l^{p()})^{\alpha()}(\Omega)$.

Then $| f |$ is a positive element of $(L^{q()}, l^{p()})^{\alpha()}(\Omega)$, from (a):

$$\| I_\gamma(| f |) \|_{q^*(), \infty, \alpha^*(), \Omega} \leq C(q(), \infty, \alpha) \| | f | \|_{q(), \infty, \alpha(), \Omega} < \infty,$$

this implies that for always every (a.e.)

$$I_\gamma(| f |)(z) = \int_\Omega | z - y |^{d(\gamma - 1)} | f(y) | < \infty, \quad z \in \Omega.$$

Therefore $I_\gamma(f)(z) = \int_\Omega | z - y |^{d(\gamma-1)} f(y) dy$ converges and verifies $\left| I_\gamma(f)(z) \right| \leq I_\gamma(| f |)(z)$. If we recapitulate, from (β_3) and (β_4), we have for any $(x, r) \in \Omega \times (0, \infty)$:

$$\left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega, dy)} \leq$$

$$\begin{cases} A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega, dy)} + 2^{-d\left(\gamma + \frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_+} - \frac{1}{q_-}\right)} \sum_{n \geq 1} 2^{-\frac{nd}{q_-^*}} \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)} & \text{if } r \in I_1 = (0, 1), \\ A \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega, dy)} + 2^{d\left(2 - \gamma - \frac{1}{q_+}\right)} r^{d\left(\frac{1}{q_-} - \frac{1}{q_+}\right)} \sum_{n \geq 1} 2^{\frac{-nd}{q_-^*}} \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)} & \text{if } r \in I_2 = [1, \infty[. \end{cases}$$

Using triangular inequality property of $\| \cdot \|_{L^{p()}(\Omega)}$, we get

$$\left\| \left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} \leq$$

$$\begin{cases} A\left\| \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} + 2^{-d\left(\gamma+\frac{1}{q_+}\right)} B(r) \sum_{n\geq 1} 2^{nd\left(\gamma-\frac{1}{q_-}\right)} \left\| \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)} \right\|_{L^{p()}(\Omega, dx)} & \text{if } r \in I_1, \\ A\left\| \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} + 2^{d\left(2-\gamma-\frac{1}{q_+}\right)} B_0(r) \sum_{n\geq 1} 2^{nd\left(\gamma-\frac{1}{q_-}\right)} \left\| \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)} \right\|_{L^{p()}(\Omega, dx)} & \text{if } r \in I_2. \end{cases}$$

But we have already proved that $B(r)$ and $B_0(r)$ are constants, respectively, equal to B and B_0, therefore

$$\left\| \left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} \leq$$

$$\begin{cases} A\left\| \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} + 2^{-d\left(\gamma+\frac{1}{q_+}\right)} B \sum_{n\geq 1} 2^{nd\left(\gamma-\frac{1}{q_-}\right)} \left\| \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)} \right\|_{L^{p()}(\Omega, dx)} & \text{if } r \in I_1, \\ A\left\| \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} + 2^{d\left(2-\gamma-\frac{1}{q_+}\right)} B_0 \sum_{n\geq 1} 2^{nd\left(\gamma-\frac{1}{q_-}\right)} \left\| \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)} \right\|_{L^{p()}(\Omega, dx)} & \text{if } r \in I_2. \end{cases} \tag{81}$$

Case $r \in I_1 = (0, 1)$.

In this case we have

$$\left\| \left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} \leq A\left\| \left\| f\chi_{J_x^{2r}} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$+ 2^{-d\left(\gamma+\frac{1}{q_+}\right)} B \sum_{n\geq 1} 2^{nd\left(\gamma-\frac{1}{q_-}\right)} \left\| \| f \|_{L^{q()}(J_x^{2^{n+1}r}, dy)} \right\|_{L^{p()}(\Omega, dx)}.$$

From (68) in case $p_+ < \infty$

$$\left\| \left\| f\chi_{J_x^r} \right\|_{L^{q()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} \leq r^{d\left(\frac{1}{q(x)}+\frac{1}{p(x)}-\frac{1}{\alpha(x)}\right)} \| f \|_{q(), p(), \alpha(), \Omega}. \tag{82}$$

From the last numbered inequality we get

$$\left\| \left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq 2^{d\left(\frac{1}{q(x)}+\frac{1}{p(x)}-\frac{1}{\alpha(x)}\right)}\left[A+2^{-d\left(\gamma+\frac{1}{q_+}\right)}B\sum_{n\geq 1}2^{nd\left(\gamma-\frac{1}{\alpha(x)}+\frac{1}{p(x)}+\frac{1}{q(x)}-\frac{1}{q_-}\right)}\right]$$

$$\times r^{d\left(\frac{1}{q(x)}+\frac{1}{p(x)}-\frac{1}{\alpha(x)}\right)}\||f|\|_{q(),\,p(),\,\alpha(),\,\Omega}$$

$$\leq 2^{d\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\alpha}\right)_+}\left[A+2^{-d\left(\gamma+\frac{1}{q_+}\right)}B\sum_{n\geq 1}2^{nd\left(\gamma-\frac{1}{\alpha(x)}+\frac{1}{p(x)}+\frac{1}{q(x)}-\frac{1}{q_-}\right)}\right]$$

$$\times r^{d\left(\frac{1}{q(x)}+\frac{1}{p(x)}-\frac{1}{\alpha(x)}\right)}\||f|\|_{q(),\,p(),\,\alpha(),\,\Omega}.$$

We have $\gamma-\frac{1}{\alpha(x)}+\frac{1}{p(x)}<0$ see the hypotheses, it is said (as in the case $q_+=\infty$) that $\frac{1}{q(x)}-\frac{1}{q_-}\leq 0,\quad x\in\Omega,$ therefore the series $\sum_{n\geq 1}2^{nd\left(\gamma-\frac{1}{\alpha(x)}+\frac{1}{p(x)}+\frac{1}{q(x)}-\frac{1}{q_-}\right)}$ converges and we let $\sum_{n\geq 1}2^{nd\left(\gamma-\frac{1}{\alpha(x)}+\frac{1}{p(x)}+\frac{1}{q(x)}-\frac{1}{q_-}\right)}=s<\infty,$ in another hand $2^{d\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\alpha}\right)_+}<\infty$ since $\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\alpha}\right)_+<\infty$, then we get

$$\left\|\left\|(I_\gamma f)\chi_{J_x^r}\right\|_{L^{q^*()}(\Omega,\,dy)}\right\|_{L^{p()}(\Omega,\,dx)}\leq 2^{d\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\alpha}\right)_+}\left[A+2^{-d\left(\gamma+\frac{1}{q_+}\right)}Bs\right]$$

$$\times r^{d\left(\frac{1}{q(x)}+\frac{1}{p(x)}-\frac{1}{\alpha(x)}\right)}\|\|f\|\|_{q(),\,p(),\,\alpha(),\,\Omega}.$$

If we let $C(q(),\ p(),\ \alpha())=2^{d\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\alpha}\right)_+}\left[A+2^{-d\left(\gamma+\frac{1}{q_+}\right)}Bs\right],$

$C(q(),\ p(),\ \alpha())<\infty$ and we get

$$\left\| \left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)} \leq C(q(),\ p(),\ \alpha())$$

$$\times\, r^{d\left(\frac{1}{q(x)} + \frac{1}{p(x)} - \frac{1}{\alpha(x)}\right)} \| f \|_{q(), p(), \alpha(), \Omega}.$$

This is equivalent to

$$r^{d\left(\frac{1}{\alpha(x)} - \frac{1}{q(x)} - \frac{1}{p(x)}\right)} \left\| \left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq C(q(),\ p(),\ \alpha()) \| f \|_{q(), p(), \alpha(), \Omega}.$$

From the hypotheses $\frac{1}{\alpha(x)} - \frac{1}{q(x)} - \frac{1}{p(x)} = \frac{1}{\alpha^*(x)} - \frac{1}{q^*(x)} - \frac{1}{p(x)}$,

therefore we get

$$r^{d\left(\frac{1}{\alpha^*(x)} - \frac{1}{q^*(x)} - \frac{1}{p(x)}\right)} \left\| \left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq C(q(),\ p(),\ \alpha()) \| f \|_{q(), p(), \alpha(), \Omega}.$$

Case $r \in I_2 = [1, \infty)$.

We proceed exactly as we did in the case Case $r \in I_1 = (0, 1)$ to get that

$$r^{d\left(\frac{1}{\alpha^*(x)} - \frac{1}{q^*(x)} - \frac{1}{p(x)}\right)} \left\| \left\| (I_\gamma f)\chi_{J_x^r} \right\|_{L^{q^*()}(\Omega, dy)} \right\|_{L^{p()}(\Omega, dx)}$$

$$\leq C(q(),\ p(),\ \alpha()) \| f \|_{q(), p(), \alpha(), \Omega}.$$

If we pass to the supremum over all $r > 0,\ x \in \Omega,$ we will get

$$\| I_\gamma f \|_{q^*(), p(), \alpha^*(), \Omega} \leq C(q(),\ p(),\ \alpha()) \| f \|_{q(), p(), \alpha(), \Omega}.$$

From Proposition 33 we will have

$$\| I_\gamma f \|_{q^*(), p(), \alpha^*(), \Omega} \leq C \| f \|_{q(), p(), \alpha(), \Omega}.$$

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