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NEW EXACT SOLUTIONS OF DOUBLE SINE-GORDON EQUATION

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Abstract

New exact rotational as well as translational invariant solutions of Double Sine-Gordon equation is obtained by the method of Lie Group Similarity Transformation. By this method a nonlinear partial differential equation is reduced to nonlinear ordinary differential equation of second order and solving exactly. This new solution is rotational as well as translational invariant, unlike known solutions that are translational invariant alone. Respective Lie Group generators and their algebra are reported.

1. Introduction

The Sine-Gordon (SG) equation is recognized as an important model in the Solid State Physics [1] and High Energy Particle Physics [2]. SG

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equation widely recognized as a completely integrable system and its Auto-Backlund transformation, Lax pairs, infinite number of conservation laws, etc. are known [3]. In a Klein-Gordon family equation with two Sine functions was introduced [1, 4] and named as Double Sine-Gordon (DSG) equation

$$u_{tt} - u_{xx} = \sin\left(\frac{u}{a}\right) - 2b \sin\left(\frac{u}{2b}\right). \tag{1.01}$$

DSG equation has several applications in nonlinear optics such as the study of the B-phase of liquid helium and the treatment of quasi-one-dimensional charge density wave condensate of organic linear conductors like TTF-TCNQ etc. [4, 5, 6].

Travelling wave solutions of DSG equation are known [7], obtained by the methods of Basic Equations [7] and Hirota's Bi-linear Operator [8]. DSG equation is widely accepted as a non-integrable system and so the travelling wave types exact solutions of this equation are not belong to soliton's family [5] of highly stable particles like solutions. In this study, we report new exact solutions of the DSG equation (1.01) obtained by the method of Lie Group Similarity Transformation [9].

2. Lie Group Similarity Transformations Method for Partial Differential Equation

Essential details of the Lie continuous point group similarity transformation method to reduce the number of independent variables of a partial differential equation (PDE) so as to obtain respective ordinary differential equation (ODE) [13] is the following. Let the given PDE in two independent variables x and t and one dependent variable u be

$$F(x, t, u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.1}$$

where $u_t, u_x, \ldots$ are all partial derivatives of dependent variables $u(x, t)$

with respect to the independent variable t and x, respectively.

When we apply a family of one parameter infinitesimal continuous point group transformations,

$$x = x + \varepsilon X(x, t, u) + O(\varepsilon^2), \tag{2.2}$$

$$t = t + \varepsilon T(x, t, u) + O(\varepsilon^2), \tag{2.3}$$

$$u = u + \varepsilon U(x, t, u) + O(\varepsilon^2), \tag{2.4}$$

we get the infinitesimals of the variables u, t and x as U, T, X, respectively and ε is an infinitesimal parameter. The derivatives of u are also transformed as

$$u_x = u_x + \varepsilon[U_x] + O(\varepsilon^2), \tag{2.5}$$

$$u_{xx} = u_{xx} + \varepsilon[U_{xx}] + O(\varepsilon^2), \tag{2.6}$$

$$u_{tt} = u_{tt} + \varepsilon[U_{tt}] + O(\varepsilon^2), \tag{2.7}$$

where $[U_x]$, $[U_{xx}]$, $[U_{tt}]$ are the infinitesimals of the derivatives u_x, u_{xx}, u_{tt}, respectively. These are called first and second extensions and are given by [16],

$$[U_x] = U_x + (U_u - X_x)u_x - X_u u_x^2 - T_x u_t - T_x u_x u_t, \tag{2.8}$$

$$\begin{aligned}[U_{xx}] = {} & U_{xx} + (2U_{xu} - X_{xx})u_x + (U_{uu} - 2X_{xu})u_x^2 - X_{uu}u_x^3 \\ & + U_u - 2X_x u_{xx} - 3X_u u_x u_{xx} - T_{xx}u_t - 2T_{xu}u_x u_t - T_{uu}u_x^2 u_t \\ & -2T_x u_{xt} - T_u u_{xx} u_t - 2T_u u_{xt} u_x,\end{aligned} \tag{2.9}$$

$$[U_{tt}] = U_{tt} + [2U_{tu} - T_{tt}]u_t - X_{tt}u_x + [U_{uu} - 2T_{uu}]u_t^2$$

$$-2X_{tu}u_xu_t - T_{uu}u_t^3 - X_{uu}u_t^2u_x + [U_u - 2T_t]u_{tt} - 2X_tu_{xt}$$

$$-3T_uu_{tt}u_t - X_uu_{tt}u_x - 3X_uu_{xt}u_t. \tag{2.10}$$

The invariant requirements of given PDE (2.1) under the set of above transformations lead to the invariant surface conditions,

$$T\frac{\partial F}{\partial t} + X\frac{\partial F}{\partial x} + U\frac{\partial F}{\partial u} + [U_x]\frac{\partial F}{\partial u_x} + [U_{tt}]\frac{\partial F}{\partial u_{tt}} + [U_{xx}]\frac{\partial F}{\partial u_{xx}} = 0. \tag{2.11}$$

On solving above invariant surface condition (2.11), the infinitesimals X, T, U can be uniquely obtained, that give the similarity group under which the given PDE (2.1) is invariant. This gives

$$T\frac{du}{dt} + X\frac{du}{dx} - \frac{du}{dU} = 0. \tag{2.12}$$

The solution of (2.12) are obtained by Lagrange's condition,

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U}. \tag{2.13}$$

This yields

$$x = x(t, C_1, C_2) \quad \text{and} \quad u = u(t, C_1, C_2), \tag{2.14}$$

where C_1 and C_2 are arbitrary integration constants and the constant C_1 plays the role of an independent variable called the similarity variable S and C_2 that of a dependent variable called the similarity solution $u(S)$ such that exact solution of given PDE, so that

$$u(x, t) = u(S). \tag{2.15}$$

On substituting (2.15) in given PDE (2.1) that reduced to an ordinary differential equation with S as independent variable and $u(S)$ as dependent variable.

3. Similarity Transformation of DSG Equation

Here we apply the Lie Group Similarity Transformation method to find exact solutions of DSG equation (1.01)

$$u_{tt} - u_{xx} = \left[\sin\left(\frac{u}{a}\right) - 2b\sin\left(\frac{u}{2b}\right)\right]. \tag{3.01}$$

The general form of (3.01) is

$$F(u, u_{xx}\ u_{tt}, x, t) = 0. \tag{3.02}$$

The invariant surface condition (2.11) gives

$$-[u_{xx}]\frac{\partial F}{\partial u_{xx}} + [u_{tt}]\frac{\partial F}{\partial u_{tt}} - u\frac{\partial\left[\sin\left(\frac{u}{a}\right) - 2b\sin\left(\frac{u}{2b}\right)\right]}{\partial u} = 0. \tag{3.03}$$

On substituting the expansions of $[u_{xx}]$, $[u_{tt}]$, and equating coefficients of different orders of derivatives of $u(x, t)$, we get the constraint equations as

$$U_{tt} - U_{xx} - \frac{U}{a}\cos\left(\frac{u}{a}\right) - U\cos\left(\frac{u}{2b}\right) = 0,$$

$$T_t = T_u = u = 0,$$

$$X_{xx} - 2U_{xu} - X_{tt} = 0,$$

$$2U_{tu} - T_{tt} + T_{xt} = 0,$$

$$T_x - X_t = 0,$$

$$T_{xu} - X_{tu} = 0,$$

$$X_u = X_x = 0. \tag{3.04}$$

On solving above set of constraints, we get

$$X = ct + v,$$

$$T = cx + k,$$

$$U = 0. \tag{3.05}$$

The Lagrange's condition (2.13)

$$\left(\frac{dt}{T}\right) = \left(\frac{dx}{X}\right) = \left(\frac{du}{U}\right), \tag{3.06}$$

gives the similarity variable $h(x, t)$ as

$$h(x, t) = \{(c/2)(x^2 - t^2) + (kx - vt) - (k^2 - v^2)/2c\}, \tag{3.07}$$

where c, k, v are arbitrary constants and c is non zero. Then the similarity solution $u[h(x, t)]$ of the hyperbolic DSG equation (3.01) is

$$u(x, t) = u[h(x, t)]. \tag{3.08}$$

On substituting (3.08) in (3.01) the DSG equation reduces to an ordinary second order differential equation

$$h\frac{d^2u(h)}{dh^2} + \frac{du(h)}{dh} = -(1/2c)\left[\sin\left(\frac{u}{a}\right) - 2b\sin\left(\frac{u}{2b}\right)\right]. \tag{3.09}$$

On solving (3.09), we get the exact solution of DSG equation (1.01) as

$$u[h(x, t)] = 4b\arctan\left[4\sqrt{h(x, t)}\right], \tag{3.10}$$

where $h(x, t)$ is eq. (3.7), k and v are arbitrary constants. Above solution is valid for all values of c except $c = 0$.

4. Discussion

Above new solution (3.10) of DSG equation (1.01) is hypothetically rotational invariant as well as translational invariant, corresponding to respective symmetries generated by three Lie Group Generators

$$X_1 = x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x},$$

$$X_2 = \frac{\partial}{\partial x},$$

$$X_3 = \frac{\partial}{\partial t}. \tag{4.01}$$

They obey the Lie algebra

$$[X_1, X_2] = -X_3,$$

$$[X_3, X_1] = X_2,$$

$$[X_1, X_2] = 0. \tag{4.02}$$

These three Lie group generators produce two different types of exact solutions of SG equations (1.01). The generator X_1 represents hyperbolic rotationally invariant solutions with respect to the infinitesimals

$$X = ct,$$

$$T = cx,$$

$$U = 0 \tag{4.03}$$

for which all the above solution (3.10) valid with $k = 0$ and $v = 0$. That is very rarely mentioned in other studies.

For the generators X_2 and X_3, we get translationally invariant solution of DSG equation (1.01), corresponding to the infinitesimals

$$T = k,$$

$$X = \omega,$$

$$U = 0, \tag{4.04}$$

for which the similarity variable is $h_1(x, t)$,

$$h_1(x, t) = (kx - vt). \tag{4.05}$$

Since parameter c can not be zero, exclusive translational invariant solution is not possible but exists only along with rotationally invariant factor. All known solutions of DSG equation are of translationally invariant without rotational invariant factors, as such above new solution belongs to a new family. Moreover, it is known that DSG equation can be converted to phi-six equation by the method of Basic equation [7] and can be obtained exact solution of phi-six equation. But above new solutions violate that procedure, and such new solutions belong to a different family is confirmed. This type of new family of solutions reported earlier for other types of Klein-Gordon equations by this author [10]. It is found that similarity Lie point group transformation method is a powerful tool for solving nonlinear PDE by converting to ODE. But method works only when given PDE is invariant under some similarity group of transformation, that need not satisfy always.

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