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NEW EXACT SOLUTIONS OF LIOUVILLE'S EQUATIONS

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Abstract

New exact rotational as well as translational invariant solutions of both parabolic and hyperbolic Liouville's equations are found out by the method of Lie point group similarity transformation. Solutions are compared with earlier studied general solutions.

1. Introduction

Liouville's equation [1, 2] is a well-studied second order nonlinear partial differential equation that appears in many fields of Mathematics and Physics. Liouville's equation describes the structure of metrics with constant Gaussian curvature which are conformal to the restrictions of

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the Euclidean metric to a two dimensional surface. In Theoretical Physics [3-10], this equation appears in the context of mean field vorticity in steady flow also as Chern-Simons in super conductivity and in Electroweak theory [11-14]. Backlund and auto-Backlund transformation, Lax Pairs are reported [2, 10, 11] for both parabolic and hyperbolic Liouville's equations. Even then both Liouville's equations are not completely integrable systems.

In this study, Lie group similarity transformation method [16] is used to find exact solutions of both parabolic and hyperbolic Liouville's equations and compared with a well-known general solution. This author already reported exact solutions of few non-linear partial differential equations by using Lie group similarity transformation [17-20].

2. Lie Group Similarity Transformation Method of Partial Differential Equation

Essential details of the Lie continuous point group similarity transformation method to reduce the number of independent variables of a partial differential equation (PDE) so as to obtain respective ordinary differential equation (ODE) [13] is the following. Let the given PDE in two independent variables x and t and one dependent variable u be

$$F(x, t, u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.1}$$

where $u_t, u_x, \ldots$ are all partial derivatives of dependent variables $u(x, t)$ with respect to the independent variable t and $x,$ respectively.

When we apply a family of one parameter infinitesimal continuous point group transformations

$$x = x + \varepsilon X(x, t, u) + O(\varepsilon^2), \tag{2.2}$$

$$t = t + \varepsilon T(x, t, u) + O(\varepsilon^2), \tag{2.3}$$

$$u = u + \varepsilon U(x, t, u) + O(\varepsilon^2), \tag{2.4}$$

we get the infinitesimals of the variables u, t and x as U, T, X, respectively and ε is an infinitesimal parameter. The derivatives of u are also transformed as

$$u_x = u_x + \varepsilon[U_x] + O(\varepsilon^2), \tag{2.5}$$

$$u_{xx} = u_{xx} + \varepsilon[U_{xx}] + O(\varepsilon^2), \tag{2.6}$$

$$u_{tt} = u_{tt} + \varepsilon[U_{tt}] + O(\varepsilon^2), \tag{2.7}$$

where $[U_x]$, $[U_{xx}]$, $[U_{tt}]$ are the infinitesimals of the derivatives u_x, u_{xx}, u_{tt}, respectively. These are called first and second extensions and that are given by [16]

$$[U_x] = U_x + (U|u - X_x)u_x - X_u u_x^2 - T_x u_t - T_x u_x u_t, \tag{2.8}$$

$$\begin{aligned}[U_{xx}] = {} & U_{xx} + (2U_{xu} - X_{xx})u_x + (U|uu - 2X_{xu})u_x^2 - X_{uu}u_x^3 \\ & + U_u - 2X_x u_{xx} - 3X_u u_x u_{xx} - T_{xx}u_t - 2T_{xu}u_x u_t - T_{uu}u_x^2 u_t \\ & -2T_x u_{xt} - T_u u_{xx} u_t - 2T_u u_{xt} u_x,\end{aligned} \tag{2.9}$$

$$\begin{aligned}[U_{tt}] = {} & U_{tt} + [2U_{tu} - T_{tt}]u_t - X_{tt}u_x + [U_{uu} - 2T_{uu}]u_t^2 \\ & - 2X_{tu}u_x u_t - T_{uu}u_t^3 - X_{uu}u_t^2 u_x + [U_u - 2T_t]u_{tt} - 2X_t u_{xt} \\ & -3T_u u_{tt} u_t - X_u u_{tt} u_x - 3X_u u_{xt} u_t.\end{aligned} \tag{2.10}$$

The invariant requirements of given PDE (2.1) under the set of above transformations lead to the invariant surface conditions

$$T\frac{\partial F}{\partial t} + X\frac{\partial F}{\partial x} + U\frac{\partial F}{\partial u} + [U_x]\frac{\partial F}{\partial u_x} + [U_{tt}]\frac{\partial F}{\partial u_{tt}} + [U_{xx}]\frac{\partial F}{\partial u_{xx}} = 0. \tag{2.11}$$

On solving above invariant surface condition (2.11), the infinitesimals X, T, U can be uniquely obtained, that give the similarity group under which the given PDE (2.1) is invariant. This gives

$$T\frac{du}{dt} + X\frac{du}{dx} - \frac{du}{dU} = 0. \tag{2.12}$$

The solution of (2.12) are obtained by Langrange's condition

$$\frac{dT}{T} = \frac{dx}{X} = \frac{du}{U}. \tag{2.13}$$

This yields

$$x = x(t, C_1, C_2) \quad \text{and} \quad u = u(t, C_1, C_2), \tag{2.14}$$

where C_1 and C_2 are arbitrary integration constants and the constant C_1 plays the role of an independent variable called the similarity variable S and C_2 that of a dependent variable called the similarity solution $u(S)$ such that exact solution of given PDE, so that

$$u(X, t) = u(S). \tag{2.15}$$

On substituting (2.15) in given PDE (2.1) gives an ordinary differential equation with S as independent variable and $u(S)$ as dependent variable.

3. Similarity Transformatin of Parabolic Liouville's Equation

Here we apply the similarity method to find exact solutions of the parabolic Liouville's equations [1, 2]

$$u_{tt} + u_{xx} = A\exp(u). \tag{3.1}$$

So general form of (3.1) is

$$F(u,\, u_{xx},\, u_{tt},\, x,\, t) = 0. \tag{3.2}$$

The invariant surface condition (2.11) gives

$$[U_{xx}]\frac{\partial F}{\partial u_{xx}} + [U_{tt}]\frac{\partial F}{\partial u_{tt}} - U\,\frac{\partial A \exp(u)}{\partial u} = 0. \tag{3.3}$$

On substituting the expansions of $[U_{xx}]$, $[U_{tt}]$, and equating coefficients of different orders of derivatives of $u(x, t)$, we get the constrained equations as

$$\begin{gathered} U_{tt} + U_{xx} - UA\exp(u) = 0, \qquad T_t = T_u = U = 0, \\ X_{xx} - 2U_{xu} - X_{tt} = 0, \qquad 2U_{tu} - T_{tt} + T_{xt} = 0, \qquad X_x X_u = 0, \\ T_x - X_t = 0, \qquad T_{xu} - X_{tu} = 0, \qquad X_x = X_u = 0. \end{gathered} \tag{3.4}$$

On solving above set of constraints, we get

$$\begin{aligned} X &= ct + w, \\ T &= -cx + k, \\ U &= 0. \end{aligned} \tag{3.5}$$

The Lagrange's condition (2.13) gives the similarity variable $p(x, t)$ as

$$p(x,\, t) = \left\{-\left(\frac{c}{2}\right)(x^2 + t^2) + (kx - wt) - (k^2 + w^2)/2c\right\}. \tag{3.6}$$

Then the similarity solution of the hyperbolic Liouville's equation (3.1) is

$$u(x,\, t) = u(p). \tag{3.7}$$

On substituting (3.6) in (3.1) the parabolic Liouville's equation reduces to an ordinary second order differential equation

$$p\,\frac{d^2u(p)}{dp^2} + \frac{du(p)}{dp} = -\frac{A}{2c}\exp(u). \tag{3.8}$$

On solving (3.8), we get

$$\exp[u(p)] = \{\,(2Bp^{n-1})\big/(B + Ap^{n})^2\,\}. \tag{3.9}$$

The arbitrary constants B, c, and k, w are non-zero and $c = -1/(nA)$. The above solution is valid for all values of n except n is zero.

On substituting the value of the similarity variable $p(x, t)$ from (3.6), we get exact solution of parabolic Liouville's equation (3.1).

4. Similarity Transformation and Exact Solution of Hyperbolic Liouville's Equation

Similarity transformation of hyperbolic part of Liouville's equation

$$u_{tt} - u_{xx} = A\exp(u). \tag{4.1}$$

As in the previous case, we get the infinitesimals

$$\begin{aligned} T &= -cx + k, \\ X &= ct + w, \\ U &= 0. \end{aligned} \tag{4.2}$$

For $U = 0$, the method of finding above infinitesimals are same for all hyperbolic Klein-Gordon equations, that this author already reported in an earlier study [20].

Similarity variable of hyperbolic Liouville's $h(x, t)$ is

$$h(x, t) = \left\{\left(\frac{c}{2}\right)(x^2 - t^2) + (kx - wt) + (k^2 - w^2)2c\right\} \tag{4.3}$$

and the similarity reduced hyperbolic Liouville's second order ordinary differential equation can be found out by substituting $u(x, t) = u(h)$, then

$$h\frac{d^2u(h)}{dh^2} + \frac{du(h)}{dh} = -\left(\frac{A}{2c}\right)\exp(u). \tag{4.4}$$

Then the exact solution of hyperbolic Liouville's equation is the solution of above equation as

$$\exp[u(x, t)] = \{(2Bh^{n-1})/(B + Ah^n)^2\}, \tag{4.5}$$

where $h(x, t)$ is the similarity variable (3.3). Parameters A, B and n are arbitrary constants and B and c, k and w are nonzero, where $c = -1/(An)$.

5. Discussion

Corresponding to the hyperbolic Liouville's equation generator of infinitesimals are the following

$$H_1 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x},$$

$$H_2 = \frac{\partial}{\partial x},$$

$$H_3 = \frac{\partial}{\partial t}. \tag{5.1}$$

They obey the Lie algebra

$$[H_1, H_2] = -H_3,$$

$$[H_3, H_1] = H_2,$$

$$[H_2, H_3] = 0. \tag{5.2}$$

These three Lie group generators produce two different types of exact solutions of both Liouville's equations. The generator H_1 represents hyperbolic rotationally invariant solutions with respect to the infinitesimals

$$X = ct,$$

$$T = cx,$$

$$U = 0 \tag{5.3}$$

for which all the above solutions are valid with $k = 0$ and $w = 0$. That very rarely mentioned in other studies.

For the generator H_2 and H_3, we get translationally invariant solutions of hyperbolic Liouville's equation corresponding to the infinitesimals

$$T = k,$$

$$X = w,$$

$$U = 0. \tag{5.4}$$

For which the similarity variable is $h_1(x, t)$,

$$h_1(x, t) = (kx - wt). \tag{5.5}$$

Similarly, for parabolic Liouville's equation, there are three Lie group generators P_1, P_2, P_3.

$$P_1 = -x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, \tag{5.6}$$

$$P_2 = \frac{\partial}{\partial x}, \tag{5.7}$$

$$P_3 = \frac{\partial}{\partial t}. \tag{5.8}$$

Respective Lie algebras are

$$[P_1, P_2] = P_3, \tag{5.9}$$

$$[P_1, P_3] = P_2, \tag{5.10}$$

$$[P_2, P_3] = 0. \tag{5.11}$$

As in the hyperbolic case, P_1 represents pure rotationally invariant solutions of Parabolic Liouville's equation, for which $k = 0$, $w = 0$. Since $c = 1/2nA$ and c is nonzero, exclusively translational invariant solution is not possible. But the translationally invariant solution exists only along with rotationally invariant solution.

Solutions of both parabolic and hyperbolic Liouville's equations can be extended to $(3+1)$ dimensions. For which the similarity variables are

$$\begin{aligned} h_1 = \{ &+ (\frac{c}{2})(x^2 + y^2 + z^2 - t^2) + (k_1 x - w_1 t) \\ &+(k_2 x - w_2 t) + (k_3 x - w_3 t) \\ &+ [(k_1^2 + k_2^2 + k_3^2) - (w_1^2 + w_2^2 + w_3^2)]/(2c) \}. \end{aligned} \tag{5.12}$$

For parabolic similarity variable in $(3+1)$ dimensions is

$$\begin{aligned} p_1 = \{ &- (\frac{c}{2})(x^2 + y^2 + z^2 + t^2) + (k_1 x - w_1 t) \\ &+(k_2 y - w_2 t) + (k_3 z - w_3 t) \\ &+ [(k_1^2 + k_2^2 + k_3^2) + (w_1^2 + w_2^2 + w_3^2)]/(2c) \}. \end{aligned} \tag{5.13}$$

Backlund as well as auto-Backlund transformations and Inverse Scattering Transformation are known for hyperbolic Liouville's equation, but not completely integrable system. So, the above solutions are not stable like solitons. Due to the rotational symmetry, these solutions may not have multi solutions like multi solitons.

The general solutions of hyperbolic Liouville's equation reported [7, 11] is

$$\exp[u(x, t)] = [2f(x).g(t)]/[\,f(x) + g(t) + z_0\,]^2. \tag{5.14}$$

Obviously, our solutions are different from this class of general solutions naturally arisen from invariant symmetry.

All even and positive values of n the denominator of solutions both parabolic and hyperbolic Liouville's equations are always nonzero when B is positive valued integration constant. Such case solutions are non-singular. Whereas, when n is odd and positive valued then the denominator may be zero and that yields singular solutions for both parabolic and hyperbolic Liouville's equations.

It is found that similarity Lie point group transformation method is a powerful tool for solving nonlinear PDE by converting to ODE. But this method works only when given PDE is invariant under some similarity group of transformation, that need not satisfy always.

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