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NEW EXACT SOLUTION OF SINE-GORDON EQUATION

B. V. BABY

3/88, Jadkal Post
Udupi District, Karnataka State
India 576233
e-mail: dr.bvbaby@yahoo.co.in

Abstract

A new exact solution of nonlinear Sine-Gordon equation is obtained by the method of Lie Group Similarity Transformation. Nonlinear partial differential equation is reduced to nonlinear ordinary differential equation of second order and solved exactly. Respective Lie Algebra and their generators are also reported.

1. Introduction

In 1876, Backlund proposed the Sine-Gordon (SG) equation [1] as a model of nonlinear pseudo spherical surface in differential geometry. He developed a method of solving SG equation, called Auto-Backlund Transformation (ABT) in which second order nonlinear partial differential equation (PDE) can be transformed into two first order

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ordinary differential equation (ODE) in which arbitrary number of exact solitons of PDE can be found out by solving the ODEs.

In theoretical Physics, SG equation is used to explain the self induced transparency phenomenon of nonlinear optics, simple model for elementary particles and also a standard field theoretical model [2]. In 1962, Perring and Skyrme [3] studied the collision process of SG equation's waves through computer simulation and confirmed their stability and their particle like property and so named Solitons [4, 5]. Generally, the term solitons are reserved for non-topological solitons of completely integrable systems [4]. Solutions of SG equations are called Kink and Breather solutions.

Two dimensional $(1+1)$ SG equation can be written as

$$u_{xt} = \sin(u) \tag{1.01}$$

and its Painleve Property (PP), Inverse Scattering Transformation (IST), Laxx Pairs are known [5] and so considered as a completely Integrable system (CIS). A well known exact soliton of SG equation [1] is

$$u_{xt} = 4\arctan[\exp(kx - vt)]. \tag{1.02}$$

This study is reporting a new exact solution of the SG equation (1.01) obtained by the method of Lie Group Similarity Transformation [6].

2. Lie Group Similarity Transformation Method of Partial Differential Equation

Essential details of the Lie continuous point group similarity transformation method to reduce the number of independent variables of a partial differential equation (PDE) so as to obtain respective ordinary differential equation (ODE) [6] is the following. Let the given PDE in two independent variables x and t and one dependent variable u be

$$F(x, t, u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.1}$$

where $u_t, u_x, \ldots$ are all partial derivatives of dependent variables $u(x, t)$ with respect to the independent variables t and x, respectively.

When we apply a family of one parameter infinitesimal continuous point group transformations,

$$x = x + \varepsilon X(x, t, u) + O(\varepsilon^2), \tag{2.2}$$

$$t = t + \varepsilon T(x, t, u) + O(\varepsilon^2), \tag{2.3}$$

$$u = u + \varepsilon U(x, t, u) + O(\varepsilon^2), \tag{2.4}$$

we get the infinitesimals of the variables u, t and x as U, T, X, respectively and ε is an infinitesimal parameter. The derivatives of u are also transformed as

$$u_x = u_x + \varepsilon[U_x] + O(\varepsilon^2), \tag{2.5}$$

$$u_{xx} = u_{xx} + \varepsilon[U_{xx}] + O(\varepsilon^2), \tag{2.6}$$

$$u_{tt} = u_{tt} + \varepsilon[U_{tt}] + O(\varepsilon^2), \tag{2.7}$$

where $[U_x]$, $[U_{xx}]$, $[U_{tt}]$ are the infinitesimals of the derivatives u_x, u_{xx}, u_{tt}, respectively. These are called first and second extensions and are given by [6],

$$[U_x] = U_x + (U_u - X_x)u_x - X_u u_x^2 - T_x u_t - T_x u_x u_t, \tag{2.8}$$

$$\begin{aligned}[U_{xx}] = {} & U_{xx} + (2U_{xu} - X_{xx})u_x + (U_{uu} - 2X_{xu})u_x^2 - X_{uu}u_x^3 \\ & + U_u - 2X_x u_{xx} - 3X_u u_x u_{xx} - T_{xx}u_t - 2T_{xu}u_x u_t - T_{uu}u_x^2 u_t \\ & -2T_x u_{xt} - T_u u_{xx}u_t - 2T_u u_{xt}u_x,\end{aligned} \tag{2.9}$$

$$\begin{aligned}[U_{tt}] = {} & U_{tt} + [2U_{tu} - T_{tt}]u_t - X_{tt}u_x + [U_{uu} - 2T_{uu}]u_t^2 \\ & - 2X_{tu}u_x u_t - T_{uu}u_t^3 - X_{uu}u_t^2 u_x + [U_u - 2T_t]u_{tt} - 2X_t u_{xt}\end{aligned}$$

$$-3T_u u_{tt} u_t - X_u u_{tt} u_x - 3X_u u_{xt} u_t. \tag{2.10}$$

The invariant requirements of given PDE (2.1) under the set of above transformations lead to the invariant surface conditions,

$$T\frac{\partial F}{\partial t} + X\frac{\partial F}{\partial x} + U\frac{\partial F}{\partial u} + [U_x]\frac{\partial F}{\partial u_x} + [U_{tt}]\frac{\partial F}{\partial u_{tt}} + [U_{xx}]\frac{\partial F}{\partial u_{xx}} = 0. \tag{2.11}$$

On solving above invariant surface condition (2.11), the infinitesimals X, T, U can be uniquely obtained, that give the similarity group under which the given PDE (2.1) is invariant. This gives

$$T\frac{du}{dt} + X\frac{du}{dx} - \frac{du}{dU} = 0. \tag{2.12}$$

The solutions of (2.12) are obtained by Lagrange's condition,

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U}. \tag{2.13}$$

This yields

$$x = x(t, C_1, C_2) \quad \text{and} \quad u = u(t, C_1, C_2), \tag{2.14}$$

where C_1 and C_2 are arbitrary integration constants and the constant C_1 plays the role of an independent variable called the similarity variable S and C_2 that of a dependent variable called the similarity solution $u(S)$ such that exact solution of given PDE, so that

$$u(x, t) = u(S). \tag{2.15}$$

On substituting (2.15) in given PDE (2.1) that reduced to an ordinary differential equation with S as independent variable and $u(S)$ as dependent variable.

3. Lie Group Similarity Transformation of SG Equation

The invariant surface condition of SG equation (1.01) is given by

$$[U_{xt}] - U\cos u = 0, \tag{3.01}$$

where $[U_{xt}]$ is the second extension of U_{xt}. Then (3.01) becomes

$$U_{xt} + (U_{xt} - T_{xt})u_t + (U_{ut} - X_{xt})u_x + (U_u - X_x - T_t)u_{xt}$$

$$- T_x u_{tt} - X_t u_{xx} + (u_{tu} - X_{xu} - T_{ut})u_x u_t - T_{xu}u_t^2$$

$$- X_{tu}u_x^2 - 2X_u u_x u_{xt} - X_u u_t u_{xx} - 2T_u u_t u_{xt}$$

$$- T_u u_x u_{tt} - X_{uu}u_t u_x^2 - T_{uu}u_x u_x^2 - U \cos u = 0. \quad (3.02)$$

On equating different partial derivatives of $u(x, t)$ as zero, we get the following set of constraint equations:

$$U_{xt} - U \cos u = 0,$$

$$U_{xu} - T_{xt} = 0,$$

$$U_u - X_x - T_t = 0,$$

$$U_{uu} - X_{xu} - T_{ut} = 0,$$

$$T_x = 0,$$

$$X_t = 0,$$

$$X_u = 0,$$

$$T_u = 0. \quad (3.03)$$

On solving above constraint equations, we get

$$X = cx + v, \quad (3.04)$$

$$T = -ct + k, \quad (3.05)$$

$$U = 0. \quad (3.06)$$

The Lagrange's conditions

$$(dx/X) = (dt/T) = (du/U) = 0, \quad (3.07)$$

leads to the following similarity variable $s(x, t)$

$$s(x,\, t) = [\, - c^2 xt + c(kx - vt) + kv\,]. \tag{3.08}$$

On substituting the similarity solution

$$u(x,\, t) = u(s), \tag{3.09}$$

in the SG equation (1.01), we get the following similarity reduced second order ordinary differential equation

$$s\,\frac{d^2 u(s)}{ds^2} = (\, -1/c^2\,)\sin u. \tag{3.10}$$

On solving (3.10), we get an exact solution

$$u(s) = 4\arctan\left(4\sqrt{s}\right). \tag{3.11}$$

Substitute eqs. (3.08) and (3.09) in eq. (3.11), we get a new exact solution of SG equation (1.01).

4. Discussion

The Similarity solution $s(x,\, t)$ of (3.08) has two sub solutions as the arbitrary constants k and v take the values zero and non zero. When k and v are non zero, we get first type of exact solution and second type of exact solution is formed when $k = v = 0.$ Even then exclusive travelling wave type [8, 9], solution as $(kx - vt)$ like well known solution (1.02) is not possible due to the constant c can not be zero. As such this types of exact solutions are belonging to different class compared to known travelling wave types solutions of solitons type [7]. Moreover, above new solutions do not obey Basic Equation method [10-13] and so not able to transform to solutions of phi-four equation. That confirms these solutions belong to a new family.

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