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MONOTONICITY OF EIGENVALUES AND CERTAIN ENTROPY FUNCTIONAL UNDER THE RICCI FLOW

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Abstract

Geometric monotone properties of the first nonzero eigenvalue of Laplacian form operator under the action of the Ricci flow in a compact n-manifold $(n \geq 2)$ are studied. We introduce certain energy functional which proves to be monotonically non-decreasing, as an application, we show that all steady breathers are gradient steady solitons, which are Ricci flat metric. The results are also extended to the case of normalized Ricci flow, where we estabilish non-existence of expanding breathers other than gradient solitons.

1. Introduction

The Ricci flow, purposely designed to solve geometrization conjecture, was introduced by Hamilton [6] in 1982. However, it gains stupendous popularity since it does not only solve geometrization conjecture but consequently provides the complete proof of the longstanding Poincare conjecturé which had been proposed over a hundred years earlier. This earned G. Perelman the Field Medal Award as listed as one of the Seven Millennium Prize Problems by the Clay Mathematics Institute in 2000. The Ricci flow has since then become a powerful tool in the hands of topologists, geometers, analysts and theoretical physicists.

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Here, we consider an n- dimensional compact manifold M^n, $n \geq 2$, on which a one parameter family of Riemannian metrics $g_{ij}(t)$, $t \in [0, \infty)$ is defined. We refer to $(M^n, g(t))$ as the solution of the Ricci flow, if it satisfies the following nonlinear evolution partial differential equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1.1}$$

written in local coordinate, where R_{ij} is the Ricci curvature tensor of the manifold. The Ricci flow is thus, a process of deforming Riemannian metric by the negative of its Ricci curvature to obtaining a nicer form. It is considered [6] together with the initial condition

$$g(0) = g_0 \tag{1.2}$$

to have a solution, at least for a short time (see also [7, 16]). This result has since been extended to non-compact case in [17].

The Ricci tensor can be linearised to obtain

$$R_{ij} = \frac{-1}{2} \Delta_g (g_{ij}) + Q_{ij}(g^{-1}, \partial g), \tag{1.3}$$

where Δ_g is the Laplace-Beltrami operator acting on manifold (M^n, g) and $Q_{ij}(g^{-1}, \partial g)$ is a lower order term, quadratic in inverse of g and its first order partial derivative. Hence, the Ricci flow equation is a heat-like (diffusion-reaction) equation.

The Laplace-Beltrami operator $\Delta_g =$ div. grad. is defined (in local coordinate) as

$$\Delta_g = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|}\, g^{ij} \partial_j), \tag{1.4}$$

where $g = g_{ij} dx^i dx^j$, $|g| =$ determinant of g and $g^{ij} = (g_{ij})^{-1}$, inverse metric. For example, in the usual Euclidean space, the Laplace-Beltrami operator is exactly the usual Laplace operator

$$\Delta = \sum_{i, j=1}^{n} \frac{\partial^2}{\partial x^i \partial x^j}, \tag{1.5}$$

where we can consider the eigenvalue problem for the Laplace

$$-\Delta u_i = \lambda_i u_i$$

and we have the sequence

$$0 \le \lambda_0 < \lambda_1 \le \lambda_2 \le .... \le \lambda_i \to \infty, \ (i \to \infty)$$

as the eigenvalues of the laplacian repeated according to their geometric multiplicities and any u_i corresponding to λ_i is the eigenfunction, the eigenspace being finite dimension. In this respect, various eigenvalue problems arise, such as

$$-\Delta u = \lambda u \text{ in } \Omega \subseteq \mathbb{R}^n, \ \partial\Omega = \phi \tag{1.6}$$

so also Dirichlet $(u = 0 \text{ on } \partial\Omega)$ and Neumann ($\frac{\partial u}{\partial N} = 0$ on $\partial\Omega$ where N is the unit normal vector exterior to the boundary of $\Omega)$ counterparts. These can easily be generalised to the Riemannian Manifold (M^n, g) with or without boundary, where the Laplace-Beltrami operator is viewed as self-adjoint operator on $L^2(M^n)$ and M has a pure point spectrum of a sequence of eigenvalues $\{\lambda_i\}_{i=1}^n$ and the eigenfunction u_i form orthonormal basis of $L^2(M^n)$ with $\|u_i\|_{L^2(M^n)} = 1$.

In this paper, we consider boundariless manifold or we easily assume the boundary is empty, in this case, the first eigenvalue is equal to zero, because, here the constant functions are nontrivial solutions of the eigenvalue problem, while the first eigenvalue is always positive, if a boundary exists. Studying the behaviours of eigenvalues of Laplacian operator is not out of place as its properties such as monotonicity, multiplicity, asymptotic etc., provide us with rich information about the topology and geometry of the underlying manifold. In the first of his three groundbreaking papers [15], G. Perelman introduces the energy functional $\mathcal{F}$ and shows that it is non-decreasing along the modified Ricci flow coupled with certain conjugate heat equation. He establishes that monotonicity of $\mathcal{F}$ implies that of the first nonzero eigenvalue of the operator $-4\Delta + R$ and applies the monotonicity to rule out nontrivial steady and expanding breathers on compact manifold. In [13], L. Ma shows that the eigenvalues of Laplace-Beltrami operator on compact domain of Riemannian manifold associated with the Ricci flow is non-decreasing but with nonnegativity assumption on the scalar curvature R and X. Cao has since extended this result to the eigenvalues of the operator $-\Delta + \frac{R}{2}$, [1]. In [2], the monotonicity of eigenvalue of $-\Delta + cR$, $c \geq \frac{1}{4}$ is established without sign assumption on the curvature operator and both compact steady and expanding Ricci breathers are trivial. In [10], a family of functional $L_i - \mathcal{F}_k$ which happens to be non-decreasing under the Ricci flow is constructed and the result extended to Rescaled Ricci flow in [11]. It turns out that the Ricci flow is a special case of Rescaled Ricci flow. More interestingly, these results can be extended to any other type Laplace operator under closed Riemannian manifold, for instance, the first eigenvalue of *p*-Laplace operator $(p \geq 2)$ with Einstein metric is monotonically non-decreasing [18], In this case, when $p = 2$, the main result coincides with that of [13]. See also [12] for results in Harmoni-Ricci flow.

Throughout this paper, we adopt Einstein summation convention, where the volume element on manifold $\sqrt{|g|}dx^i = d\mu$, metric $g(\partial_i, \partial_j) = g_{ij}$, where $\partial_i = \frac{\partial}{\partial x^i}$ are the components of the metric. The Levi-Civita connection is defined by $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k$ while its Christoffel's symbols are given by $\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$. R_{ij} and R are the Ricci and scalar curvature tensors respectively, where $R = g^{ij} R_{ij}$, the trace of Ricci tensor. The contracted second Bianchi identity is given as $g^{ij}\nabla_i R_{jk} = \frac{1}{2}\nabla_k R$ and the inner product $\langle p, q \rangle := \int_{M^n} g^{ik} g^{jl} p_{ij} q_{kl} d\mu_g$. We sometimes write M instead of M^n to mean Manifold of dimension $= n$ without fear of confusion.

We note that the geometric quantities associated with the underlying manifold evolve as the manifold itself evolves under the Ricci flow, for instance, we consider the evolution of those quantities that will be directly useful in the subsequent sections.

Lemma 1. *If a one-parameter family of metric* $g(t)$ *solves the Ricci flow* (1.1), *then*, *the inverse metric*, *the Christoffel's symbols*, *the volume element*, *the scalar curvature and Laplace-Beltrami operator evolve as follows*

$$\frac{\partial}{\partial t} g^{ij} = 2g^{ik} g^{jl} R_{kl}, \qquad \frac{\partial}{\partial t} \Gamma_{ij}^{k} = g^{kl}(\partial_i R_{jl} + \partial_j R_{il} - \partial_l R_{ij}),$$

$$\frac{\partial}{\partial t} d\mu = \frac{1}{2} g^{ij}\left(\frac{\partial}{\partial t} g_{ij}\right) d\mu = -R d\mu, \qquad \frac{\partial}{\partial t} R = \Delta R + 2|R_{ij}|^2,$$

$$\frac{\partial}{\partial t} \Delta_{g(t)} = 2R_{ij} \cdot \nabla^i \nabla^j.$$

(see [4, 6]).

The rest of the paper follows; in Section 2, we discuss some classical energy functionals and lay emphasis on Perelman entropy. In Section 3, we construct a new family of entropy functionals which proves to be monotonically non-decreasing. We also discuss the monotonic properties of eigenvalues under the Ricci flow, while the results are extended to the case of normalized flow in the last section.

2. Classical Energy Functionals

2.1. Total scalar curvature

We define the total scalar curvature on a closed manifold $(M^n, g(t))$ as

$$\frac{\partial}{\partial t} \int_M R d\mu = \int_M \left(\frac{1}{2}(tr_g h)R - h^{ij} R_{ij}\right) d\mu \tag{2.1}$$

which coincides with the first variation of the classical Einstein Hilbert functional $\mathcal{H}$

$$\mathcal{H}(g_{ij}) = \int_M R d\mu \tag{2.2}$$

considering the following variation formuals

$$\frac{\partial g_{ij}}{\partial t} = h_{ij} \; \frac{\partial R}{\partial t} = -\Delta(tr_g h) + \delta^2 h - \langle h, \text{Ric}\rangle,$$

where $\delta^2 h = g^{ij} g^{pq} \nabla_j \nabla_q h_{ip}$ and $\langle h, \text{Ric}\rangle = g^{ij} g^{kl} h_{ik} R_{jl}$. Specifically,

$$\frac{\partial}{\partial t} \mathcal{H}(g_{ij}) = \int_M \left[-\Delta(tr_g h) + \delta^2 h - \langle h, \text{Ric}\rangle + \frac{R}{2} trh\right] d\mu$$

$$= \int_M \left[\frac{R}{2}\langle g, h\rangle - \langle h, \text{Ric}\rangle\right] d\mu$$

$$= \int_M h^{ij}\left(\frac{R}{2} g_{ij} - R_{ij}\right) d\mu,$$

where $G_{ij} = R_{ij} - \frac{R}{2} g_{ij}$ is the Einstein tensor. Then, we have

$$\frac{\partial}{\partial t} \mathcal{H}(g_{ij}) = \int_M - h^{ij} G_{ij} d\mu = \int_M \langle h, \nabla \mathcal{H}(g) \rangle d\mu$$

and then obtain

$$\frac{\partial}{\partial t} g = \nabla \mathcal{H}(g) \tag{2.3}$$

as the gradient flow of $\mathcal{H}(g)$.

And for the gradient flow of the Einstein-Hilbert functional we have

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + Rg_{ij} = -2G_{ij} \tag{2.4}$$

which is not parabolic, even weakly, thus, we can not readily establish its solution, even for a short time. We note that the weakly part of (3.4) coincides with the Ricci flow, while the remaining term arises from the presence of the volume element $d\mu$ which itself is time evolving and we shall however deal with this in Section 3.

Remark 2. We call g stationery of $\mathcal{H}(g)$ if $\delta\mathcal{H}(g) = 0$ for all $h \in \Gamma(S^2 T^8 M)$. Since $G_{ij} = G_{ji}$, then $G_{ij} = 0$ on M. Taking the trace, we have

$$0 \equiv G = \frac{2-n}{2} R. \tag{2.5}$$

So in dimension $n \neq 2$, this implies $R \equiv 0$ on M and therefore $\mathrm{Ric} \equiv 0$ on M (Ricci flat manifold), then the functional becomes invariant under deformations.

It is now clear that the Ricci flow is not a gradient flow of a functional over the space of smooth metric but can be formulated as a gradient-like flow. The key to achieving this is to look for functionals whose critical points are Ricci solitons, this is contained in the work of Perelman [15] as we briefly survey in the next section.

2.2. The Perelman's energy functional

Let $(M^n, g_{ij}(t))$ be a closed manifold for a Riemannian metric $g_{ij}(t)$ and a smooth function f on M^n, Perelman's Energy functional [15] on pairs (g_{ij}, f) is defined by

$$\mathcal{F}(g_{ij}(t), f) = \int_{M^n} (R + |\nabla f|^2) e^{-f} d\mu. \tag{2.6}$$

The introduction of function f has embedded the space of Riemmanian metric in a larger space (see also [9, 3]). Taking the smooth variations of metric g and f as $\delta g_{ij} = h_{ij}$ and $\delta f = K$, where $H = \frac{1}{2} tr_g h_{ij}$, we have the following variation formula

$$\delta\mathcal{F}(g_{ij}(t), f) = \int_M \Big[-\Delta H + \nabla_i \nabla_j h_{ij} - h_{ij} R_{ij} + 2\langle \nabla f, \nabla K \rangle - h_{ij} \nabla_i f \nabla_j f$$

$$+ (R + |\nabla f|^2)\left(\frac{H}{2} - K\right)\Big] e^{-f} d\mu. \tag{2.7}$$

Applying integration by parts to some terms in (3.7), we obtain

$$\delta\mathcal{F}(g_{ij}(t), f)$$

$$= \int_M \left[-h_{ij}(R_{ij} + \nabla_i \nabla_j f) + (2\Delta f - |\nabla f|^2 + R)\left(\frac{H}{2} - K\right)\right] e^{-f} d\mu. \tag{2.8}$$

Keeping the volume measure static, i.e., letting $e^{-f} d\mu =: dm$, we have $H = 2K$, and we can then consider the L^2 -gradient flow

$$h_{ij} = \frac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i \nabla_j f)$$

of the functional

$$\mathcal{F}^m = \int_M (R + |\nabla f|^2) dm, \tag{2.9}$$

whenever this flow exists, it is the Ricci flow modified by diffeomorphism generated by the gradient of f and it is equivalent to the Ricci flow.

Perelman proved that the $\mathcal{F}$ -energy functional is monotonically non-decreasing under the following coupled system of modified Ricci flow and backward heat equation

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i \nabla_j f), \\ \dfrac{\partial f}{\partial t} = -\Delta f - R. \end{cases} \tag{2.10}$$

Precisely

$$\frac{d\mathcal{F}}{dt} = 2\int_{M^n} |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu \geq 0. \tag{2.11}$$

Now modulo out the action of diffeomorphism invariance from the system (2.10), the monotonicity formulae (2.11) still holds for the following couple system

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2R_{ij}, \\ \dfrac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R. \end{cases} . \tag{2.12}$$

In application, we usually solve the Ricci flow forward in time and solve the conjugate heat equation backward in time to obtain the solution of the coupled system. To develop a controlled quantity for the Ricci flow, define

$$\lambda(g_{ij}) = \inf\left\{\mathcal{F}(g_{ij}, f) : f \in C_c^\infty(M), \int_M e^{-f} d\mu = 1\right\}, \tag{2.13}$$

where the infimum is taken over all smooth functions *f*. Setting $e^{-f} =: u,$ then the functional $\mathcal{F}$ is written as

$$\mathcal{F} = \int_{M^n} (Ru^2 + 4|\nabla u|^2 d\mu, \text{ with } \int_M u^2 d\mu = 1. \tag{2.14}$$

Then $\lambda(g)$ is the first nonzero (least) eigenvalue of the self adjoint modified operator $-4\Delta + R.$ and the non-decreasing monotonicity of $\mathcal{F}$ implies that of $\lambda.$ As an application, Perelman was able to rule out the existence of nontrivial steady or expanding Ricci breathers on closed manifolds.

Proposotion 3 ([9, 15])**.** Let $g_{ij}(t)$ be a solution of the Ricci flow and $\varphi_t : M \to M$ is any diffeomorphism on *M*, then

$$\lambda(\varphi_t^* g_{ij}) = \lambda(g_{ij})$$

and $\lambda(g_{ij})$ is monotonically non-decreasing. However, a steady breather is necessarily a steady gradient soliton.

3. A New Family of Entropy Functionals

3.1. $\mathcal{B}$-energy functional

To circumvent the difficulty encounter under Einstein-Hilbert functional, we can replace the evolving measure $d\mu$ by some static measure dm and define a new functional

$$\mathcal{B} = \int_M R dm.$$

Now

$$\frac{d\mathcal{B}}{dt} = \int_M \left[(\Delta R + 2|R_{ij}|^2) dm + R \frac{\partial}{\partial t} dm \right] \tag{3.1}$$

since dm is static, we cannot apply divergence theorem which applies to evolving measure, we then set $dm := e^{-f} d\mu$ for scalar function $f : M \to \mathbb{R}$ and therefore obtain

$$\begin{aligned} \frac{d\mathcal{B}}{dt} &= \int_M (\Delta R + 2|R_{ij}|^2 - R \frac{\partial}{\partial t} f - R^2) e^{-f} d\mu \\ &= \int_M \left[(\Delta R + 2|R_{ij}| - R(-\Delta f + |\nabla f|^2 - R) - R^2 \right] e^{-f} d\mu \\ &= 2\int_M |R_{ij}|^2 e^{-f} d\mu + \int_M \Delta \operatorname{Re}^{-f} d\mu + \int_M R(\Delta f - |\nabla f|^2 e^{-f} d\mu \\ &= 2\int_M |R_{ij}|^2 e^{-f} d\mu, \end{aligned}$$

where $\int_M \Delta \operatorname{Re}^{-f} d\mu = \int_M R\Delta e^{-f} d\mu = \int_M R(-\Delta f + |\Delta f|^2) e^{-f} d\mu$ by using integration by parts.

Then, even by inspection, if the modified Ricci flow $\frac{\partial g_{ij}}{\partial t} = -2R_{ij} - 2\nabla_i \nabla_j f$ is considered as an L^2-gradient flow of Perelman's energy functional $\mathcal{F}$, we can easily conclude that the Ricci flow $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$ is also an L^2-gradient flow of our functional $\mathcal{B}$.

Theorem 4. *Let* $(M^n, g_{ij}(t)), t \in [0, T)$ *be a solution of the Ricci flow, then*

$$\frac{d}{dt}\mathcal{B}(g_{ij}, f) = 2\int_M |R_{ij}|^2 e^{-f} d\mu, \tag{3.2}$$

where $f = \log\left(\frac{d\mu}{dm}\right)$ *and satisfies*

$$\frac{\partial}{\partial t} f = -\Delta f + |\nabla f|^2 - R. \tag{3.3}$$

In particular $\mathcal{B}(g_{ij}, f)$ *is monotonically non-decreasing in time without sign assumption on he curvature operator and the monotonicity is strict unless* $R_{ij} \equiv 0$. *Moreover, there is no nontrivial Ricci breather except gradient steady Ricci soliton, which is necessarily flat.*

Proof.

$$\frac{\partial}{\partial t} f = \frac{1}{2} tr\left(\frac{\partial}{\partial t} g_{ij}\right) = \frac{1}{2} g^{ij}\left[-2(R_{ij} + \nabla_i \nabla_j f)\right] = -R - \Delta f$$

modulo the diffeomorphism out of $\frac{\partial}{\partial t} g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f)$,

$$\frac{\partial}{\partial t} f = -\Delta f + |\nabla f|^2 - R.$$

Then,

$$\frac{d}{dt}\mathcal{B}(g_{ij}, f) = 2\int_M |R_{ij}|^2 e^{-f} d\mu \geq 0,$$

where equality holds if and on if $R_{ij} \equiv 0$ which implies that $(M^n, g_{ij}(t))$ is Ricci flat (steady gradient Ricci soliton).

3.2. The entropy formula and its monotonicity

In this section, we construct a new entropy formula for the Ricci flow, the motivations for this are the behaviours of our functional $\mathcal{B}$ (Theorem 8) under the Ricci flow modulo diffeomorphism invariance and the classical results for Dirichlet energy functional for heat flow on Riemannian manifolds. It is well known that a typical heat equation for a function $f : M^n \times [0, \infty) \to \mathbb{R}$ on an n-compact manifold M (possibly without boundary) is a gradient flow for the classical Dirichlet energy functional

$$E(f) := \frac{1}{2}\int_{M^n} |\nabla f|^2 d\mu. \tag{3.4}$$

Since there is natural L^2-inner product on S^2T^*M. An application of this is that any periodic (breather) solutions to the heat equation are harmonic function which in fact must be constant in M. The Li-Yau gradient estimate for the heat equation on complete Riemannian manifold suggests an entropy formula which was derived in [14] but proved to be monotone decreasing with non-negativity condition on Ricci curvature.

Definition 5. Let (M^n, g) be a closed n-dimensional Riemannian Manifold, $f : M^n \to \mathbb{R}$ be a smooth function on M^n, define a functional on pairs (g_{ij}, f) by

$$\mathcal{F}_{\mathcal{B}} = \int_M \left(\frac{1}{2}|\nabla f|^2 + R\right) dm, \tag{3.5}$$

where $dm := e^{-f} d\mu$.

This is a variant of Perelman's energy functional $\mathcal{F}$, though expected to behave in similar manner, it differs from the later by the introduction of constant $\frac{1}{2}$.

Let $\delta g_{ij} = h_{ij}$ and $\delta f = K$, where $H = \frac{1}{2} tr_g h_{ij}$, we have the first variation of $\mathcal{F}_{\mathcal{B}}$ as

$$\delta\mathcal{F}_{\mathcal{B}} = \int_M - h_{ij}(R_{ij} + \nabla_i \nabla_j f + |\nabla f|^2). \tag{3.6}$$

The coupled modified Ricci flow equation with a backward heat equation

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i \nabla_j f + |\nabla f|^2), \\ \dfrac{\partial f}{\partial t} = -R - \Delta f + |\nabla f|^2 g^{ij} \end{cases} \tag{3.7}$$

is a gradient flow. Conjugating away the infinitesimal diffeomorphism converts (3.7) to (2.12).

Theorem 6. *Let* $g_{ij}(t))$ *and f solves the system* (2.12) *in the interval* $[0, T)$ *then*,

$$\frac{d}{dt}\mathcal{F}_{\mathcal{B}}(g_{ij}, f) = \int_M |R_{ij} + \nabla_i \nabla_j f|^2 dm + \int_M |R_{ij}|^2 dm. \tag{3.8}$$

Showing that $\mathcal{F}_{\mathcal{B}}(g_{ij}, f)$ *is monotonically non-decreasing in time*, *however*, *the monotonicity is strict*, *unless* $R_{ij} \equiv 0$ *and f is a constant*.

Proof.

$$\mathcal{F}_{\mathcal{B}} = \int_M \left(\frac{1}{2}|\nabla f|^2 + R\right)e^{-f} d\mu = \frac{1}{2}\int_M (|\nabla f|^2 + R)e^{-f} d\mu + \frac{1}{2}\int_M \mathrm{Re}^{-f}\, d\mu$$

therefore

$$\frac{d}{dt}\mathcal{F}_{\mathcal{B}}(g_{ij}, f) = \frac{1}{2}\frac{d}{dt}\mathcal{F} + \frac{1}{2}\frac{d}{dt}\mathcal{B}.$$

The result then follows.

Definition 7. Let (M^n, g) be a closed n-dimensional Riemannian Manifold, define a family of functional $\mathcal{F}_{\mathcal{B}C}$ as

$$\mathcal{F}_{\mathcal{B}C} = \int_M (|\nabla f|^2 + 2CR)dm, \tag{3.9}$$

where $C > 0$, $C \in \mathbb{R}$. When $C = \frac{1}{2}$, this is Perelman's $\mathcal{F}$ functional [15], $C = 1$ is a specific case we consider and $C = \frac{1}{2}k$, $k \geq 1$, we have Li - $\mathcal{F}_k$ family [11].

Remark 8. Our functional $\mathcal{F}_{\mathcal{B}C}$ is a variant of Perelman functional which uses certain multiple of Dirichlet energy. Their monotonicities are consistent with each other. Ricci flow cannot be viewed as L^2-gradient flow of a certain family of $\mathcal{F}_k$ constructed in [10].

Theorem 9. *Let* $(M^n, g_{ij}(t))$, $t \in [0, T)$ *be a solution of the Ricci flow and f evolves by a conjugate heat equation or satisfies* $e^{-f} = \frac{dm}{d\mu}$, *then*, *under the coupled system* (3.12), $\mathcal{F}_{\mathcal{B}C}$ *is monotonically non-decreasing*. *In particular*, *we* *have*

$$\frac{d}{dt}\mathcal{F}_{\mathcal{B}C} = 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 dm + 2(2C-1)\int_M |R_{ij}|^2 dm \geq 0. \qquad (3.10)$$

Moreover, the monotonicity is strict unless $R_{ij} + \nabla_i\nabla_j f \equiv 0$, i.e., *there is no nontrivial breathers except steady gradient Ricci soliton and the gradient function f is constant.*

This shows that all steady breathers are gradient steady Ricci soliton with $f = 0$. An example of this is Hamilton cigar soliton (2- dimensional $\mathbb{R}^2$) with conformal metric $ds^2 = \frac{dx^2 + dy^2}{1 + x^2 y^2}$ and the gradient function $f = \log\sqrt{1 + x^2 + y^2}$.

Proof. The proof follows a direct computation based on the previous results.

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_{\mathcal{B}C} &= \frac{d}{dt}\int_M (|\nabla f|^2 + 2CR)dm \\ &= \frac{d}{dt}\left(\int_M (|\nabla f|^2 + R)dm + (2C-1)\int_M |R_{ij}|^2 dm\right. \\ &= \frac{d}{dt}\mathcal{F} + (2C-1)\frac{d}{dt}\mathcal{B}.\end{aligned}$$

Equation (3.10) follows at once.

$$\frac{d}{dt}\mathcal{F}_{\mathcal{B}C}(g_{ij}, f) \equiv 0$$

if and only if $R_{ij} \equiv 0$ and *f* is a constant.

3.3. Eigenvalues and their monotonicity

In this section, we discuss the monotonicity properties of the least eigenvalue of a self adjoint modified operator $-2\Delta + CR$ that occurs in our functional. This is important as it enables us gain controlled geometric quantity for the Ricci flow.

$$\mu_C(g_{ij}) = \inf\left\{\mathcal{F}_{\mathcal{B}C}(g_{ij}, f) : f \in C_c^\infty(M), \int_M e^{-f} d\mu = 1\right\}, \qquad (3.11)$$

where the infimum is taken over all smooth functions *f*. The normalization

$\int_M e^{-f} d\mu = 1$ makes dm a probability measure and ensures a meaningful infimum. Setting $e^{-f} =: u^2$, then, the functional $\mathcal{F}_{\mathcal{B}C}$ can be written in terms of u as

$$\mathcal{F}_{\mathcal{B}C} = \int_M (2|\nabla u|^2 + CRu^2)d\mu, \text{ with } \int_M u^2 d\mu = 1. \tag{3.12}$$

Then $\mu_C(g_{ij}) = \lambda_1(-2\Delta + CR)$ is the least eigenvalue of the self-adjoint modified operator $(-2\Delta + CR)$. Let v be the corresponding eigenfuction, then, we have

$$-2\Delta v + CRv = \mu_C(g_{ij})v$$

and $f_C = -2\log v$ is a minimiser of

$$\mu_C(g_{ij}) = \mathcal{F}_{\mathcal{B}C}(g_{ij}, f_C).$$

By standard existence and regularity theories, the minimising sequence always exists.

Theorem 10. *Let $(M^n, g_{ij}(t))$, $t \in [0, T)$ be a solution of the Ricci flow, then, the least eigenvalue $\mu_C(g_{ij})$ of $(-\Delta + CR)$ is diffeomorphism invariance and non-decreasing. The monotonicity is strict unless the metric is a steady gradient soliton.*

Proof. Let $\phi : M \to M$ be a one parameter family of diffeomorphism. For any diffeomorphism $\phi(t)$ we have

$$\mathcal{F}_{\mathcal{B}C}(\phi_t^* g_{ij}, f \circ \phi) = \mathcal{F}_{\mathcal{B}C}(g_{ij}, f)$$

then

$$\mu_C(\phi_t^* g_{ij}(t)) = \mathcal{F}_{\mathcal{B}C}(\phi_t^* g_{ij}, f_C) = \phi_t^* \mathcal{F}_{\mathcal{B}C}(g_{ij}(t), f_C)$$
$$= \mathcal{F}_{\mathcal{B}C}(g_{ij}(t), f_C) = \mu_C(g_{ij}(t)).$$

Solving the backward heat equation at any time $t \in [0, T)$ with initial condition $f(t_0) = f_0$, we know that f_0 is a minimizer with $\int_M e^{-f} d\mu = 1$. So our solution $f(t), t < t_0$ which satisfies $e^{-f} d\mu$ is also a minimizer. By Theorem 9, $\mathcal{F}_{\mathcal{B}C}(g_{ij}, f_c)$ is non-decreasing, then we have

$$\mu_C(g_{ij}(t)) = \inf \mathcal{F}_{\mathcal{B}C}(g_{ij}(t), f(t)) \le \inf \mathcal{F}_{\mathcal{B}C}(g_{ij}(t_0), f(t_0)) = \mu_C(g_{ij}(t_0)).$$

Thus, μ_C is non-decreasing under the coupled Ricci flow.

Suppose the monotonicity is not strict, then, for any times $t_1, t_2, t_1 < t_2$, the solution $g_{ij}(t)$ of the Ricci flow satisfies

$$\mu_C(g_{ij}(t_1)) = \mu_C(g_{ij}(t_2)).$$

If $f(t_1)$ is a minimizer of $\mathcal{F}_{\mathcal{B}C}(g_{ij}(t), f_c$ at time t_1, so that

$$\mu_C(g_{ij}(t_1)) = \mathcal{F}_{\mathcal{B}C}(g_{ij}(t_1),\ f(t_1)).$$

But by the monotonicity of $\mathcal{F}_{\mathcal{B}C}$

$$\mathcal{F}_{\mathcal{B}C}(g_{ij}(t_1)),\ f(t_1)) \leq \mathcal{F}_{\mathcal{B}C}(g_{ij}(t_2),\ f(t_2)).$$

$$= \mu_C(g_{ij}(t_2)).$$

This contradiction implies that

$$\frac{d}{dt}\mu_C(g_{ij}(t)) \geq 0.$$

Hence, the last part of the theorem follows clearly.

We conclude this section with the fact that there is no compact steady Ricci breather other than Ricci flat metric, this is due to the diffeomorphism invariance of the eigenvalues ([1, Theorem 3], [6], [8], [10, Theorem 55] [15]).

4. Monotonicity Formula under the Normalized Ricci Flow

The normalized Ricci flow is given [6] as

$$\frac{\partial \tilde{g}_{ij}}{\partial t} g_{ij} = -2\tilde{R}_{ij} + \frac{2}{n} r\tilde{g}_{ij}, \tag{4.1}$$

where $r = (Vol_{\tilde{g}})^{-1} \int_M \tilde{R} d\tilde{\mu}$ is a constant, the average of the scalar curvature of M, and $Vol_{\tilde{g}} = \int_M d\tilde{\mu}$. The factor r appearing in (4.1) keeps the volume of the manifold constant. Here, we extend the results from previous sections (Theorems 4, 6, 9 and 10) to the case of the normalized Ricci flow. We recall that there is a bijection between the Ricci flow (1.1) and the NRF (4.1), if we choose a normalization factor

$\phi := \phi(t)$ with $\phi(0) = 1$ such that $\tilde{g}(t) = \phi(t)g(t)$ and define a time scale $\tilde{t} = \int_0^t \phi(\tau)d\tau,$ then $\tilde{g}(t)$ solves (4.1) whenever $g(t)$ solves (1.1)

Remark 11. If $r = 0,$ all the properties of the Ricci flow (1.1) including the monotonicity of the eigenvalues of Laplacian hold without further alteration.

The following shows how geometric quantities evolve under the normalized Ricci flow;

Lemma 12. *Suppose* $\tilde{g}(t)$ *solves* (4.1), *we have*

$$\frac{\partial}{\partial t}\tilde{g}^{ij} = 2\left(\tilde{R}^{ij} - \frac{r}{n}\tilde{g}^{ij}\right), \quad \frac{\partial}{\partial t}\tilde{R} = \tilde{\Delta}\tilde{R} + 2|\tilde{R}_{ij}|^2 - \frac{2r}{n}\tilde{R},$$

$$\frac{\partial}{\partial t}d\tilde{\mu} = (r - \tilde{R})d\tilde{\mu}, \quad \frac{\partial}{\partial t}\tilde{\Delta}_{\tilde{g}} = 2\tilde{R}^{ij}\cdot\tilde{\nabla}_i\tilde{\nabla}_j - \frac{2r}{n}\tilde{\Delta}_{\tilde{g}}.$$

4.1. Monotonicity of the entropy formula

In this section, we extend some results in Section 3 to the case of NRF. Define a modified Normalized Ricci flow by

$$\frac{\partial\tilde{g}_{ij}}{\partial t} = -2\tilde{R}_{ij} + \frac{2}{n}r\tilde{g}_{ij} - 2\tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}$$

and $-\tilde{f} = \log\left(\frac{dm}{d\tilde{\mu}}\right),$ i.e.,

$$\frac{\partial\tilde{f}}{\partial t} = \frac{1}{2}tr_g\frac{\partial}{\partial t}\tilde{g}_{ij} = \frac{1}{2}\tilde{g}^{ij}\left(-2\tilde{R}_{ij} + \frac{2}{n}rg_{ij} - 2\tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}\right)$$

$$= -\tilde{R} + r - \tilde{\Delta}\tilde{f}.$$

It is however clear that the coupled system

$$\begin{cases} \dfrac{\partial\tilde{g}_{ij}}{\partial t} = -2\left(\tilde{R}_{ij} - \dfrac{r}{n}\tilde{g}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}\right), \\ \dfrac{\partial\tilde{f}}{\partial t} = -\tilde{\Delta}\tilde{f} - \tilde{R} + r \end{cases} \tag{4.2}$$

is equivalent to

$$\begin{cases} \dfrac{\partial\tilde{g}_{ij}}{\partial t} = -2\tilde{R}_{ij} + \dfrac{2}{n}r\tilde{g}_{ij}, \\ \dfrac{\partial\tilde{f}}{\partial t} = -\tilde{\Delta}\tilde{f} + |\nabla f|^2 - \tilde{R} + r. \end{cases} \tag{4.3}$$

Now using Perelman's energy functional $\widetilde{\mathcal{F}} = \phi\mathcal{F}$, i.e., $\widetilde{\mathcal{F}} = \int_M (|\widetilde{\nabla}\widetilde{f}|^2 + \widetilde{R})e^{-\widetilde{f}} d\widetilde{\mu}$, we have

$$\frac{d\widetilde{\mathcal{F}}}{dt} = 2\int_M |\widetilde{R}_{ij} + \widetilde{\nabla}_i\widetilde{\nabla}_j\widetilde{f}|^2 e^{-\widetilde{f}} d\widetilde{\mu} - \frac{2r}{n}\int_M \widetilde{g}^{ij}(\widetilde{\nabla}_i\widetilde{\nabla}_j\widetilde{f} + \widetilde{R}_{ij})e^{-\widetilde{f}} d\widetilde{\mu}$$

$$= 2\int_M |\widetilde{R}_{ij} + \widetilde{\nabla}_i\widetilde{\nabla}_j\widetilde{f}|^2 e^{-\widetilde{f}} d\widetilde{\mu} - \frac{2r}{n}\widetilde{\mathcal{F}}.$$

So, $\frac{d\widetilde{\mathcal{F}}}{dt} \geq 0$ whenever $r \leq 0$. Thus we have proved the following;

Theorem 13. *Let* $(\widetilde{g}_{ij}, \widetilde{f})$ solves (4.3) *in the interval* $[0, T)$, *then*

$$\frac{d\widetilde{\mathcal{F}}}{dt} = 2\int_M |\widetilde{R}_{ij} + \widetilde{\nabla}_i\widetilde{\nabla}_j\widetilde{f}|^2 e^{-\widetilde{f}} d\widetilde{\mu} - \frac{2r}{n}\widetilde{\mathcal{F}} \geq 0, \tag{4.4}$$

when $r \leq 0$.

Theorem 14. *Suppose* $\widetilde{g}_{ij}(t)$ *is a solution of* (4.1) *and we define energy functional*

$$\widetilde{\mathcal{B}} = B(\widetilde{g}_{ij}, \widetilde{f}) = \int_M \widetilde{\mathrm{R}}\mathrm{e}^{-\widetilde{f}}\, d\widetilde{\mu} \tag{4.5}$$

then,

$$\frac{d\widetilde{\mathcal{B}}}{dt} = 2\int_M |\widetilde{R}_{ij}|^2 e^{-\widetilde{f}} d\widetilde{\mu} - \frac{2r}{n}\widetilde{\mathcal{B}}. \tag{4.6}$$

And $\widetilde{\mathcal{B}}$ *is non-decreasing whenever* $r \leq 0$, *where* $-\widetilde{f} := \log\left(\frac{dm}{d\widetilde{\mu}}\right)$. *The monotonicity is strict unless we are on Ricci flat metric*.

Proof.

$$\frac{d\widetilde{\mathcal{B}}}{dt} = 2\int_M \frac{\partial\widetilde{R}}{\partial t} - \widetilde{R}\frac{\partial\widetilde{f}}{\partial t} - \widetilde{R}(r - \widetilde{R})e^{-\widetilde{f}} d\widetilde{\mu}$$

$$= 2\int_M \left[\widetilde{\Delta}\widetilde{R} + 2|\widetilde{R}_{ij}|^2 - \frac{2r}{n}\widetilde{R} - \widetilde{R}(-\widetilde{\Delta}\widetilde{f} + |\widetilde{\nabla}\widetilde{f}|^2 - \widetilde{R} + r) - \widetilde{R}(r - \widetilde{R})\right]e^{-\widetilde{f}} d\widetilde{\mu}$$

$$= 2\int_M |\tilde{R}_{ij}|^2 e^{-\tilde{f}} d\tilde{\mu} - \frac{2r}{n}\int_M \tilde{\text{R}}\text{e}^{-\tilde{f}}\ d\tilde{\mu}.$$

Therefore our new entropy functional (3.9) implies

$$\tilde{\mathcal{F}}_{BC} = \mathcal{F}_{BC}(\tilde{g}_{ij}, \tilde{f}) = \int_M (|\tilde{\nabla}\tilde{f}|^2 + 2C\tilde{R})e^{-\tilde{f}} d\tilde{\mu} = \tilde{\mathcal{F}} + (2C-1)\tilde{\mathcal{B}}. \quad (4.7)$$

Hence

$$\frac{d}{dt}\tilde{\mathcal{F}}_{BC} = 2\int_M |\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2 e^{-\tilde{f}} d\tilde{\mu} + 2(2C-1)$$

$$\times \int_M |\tilde{R}_{ij}|^2 e^{-\tilde{f}} d\tilde{\mu} - \frac{2r}{n}\tilde{\mathcal{F}} - 2(2C-1)\frac{r}{n}\tilde{\mathcal{B}}$$

$$= 2\int_M |\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2 e^{-\tilde{f}} d\tilde{\mu} + 2(2C-1)\int_M |\tilde{R}_{ij}|^2 e^{-\tilde{f}} d\tilde{\mu} - \frac{2r}{n}\tilde{\mathcal{F}}_{BC} \quad (4.8)$$

≥ 0 (where $r \leq 0$).

Theorem 15. *Let* $\tilde{g}_{ij}(t)$, $t \in [0, T)$ *solves the normalized Ricci flow and* $\tilde{f}$ *the conjugate heat equation under the coupled system* (4.3). *Then*, $\tilde{\mathcal{F}}_{BC}$ is *monotonically non-decreasing when* $r \leq 0$. *Moreso*, *if* $r = 0$, *then the monotonicity is strict*, *unless the metric* $\tilde{g}_{ij}(t)$ *is Ricci flat and* $\tilde{f}$ *is a constant function*.

Our monotonicity formula does not classify the metric if *r* is negative, though this is not difficult to achieve, we need a little modification (This case is done by J. Li [Theorem 1.4 11])

4.2. Monotonicity of the least eigenvalue under the NRF

Let $g(t)$ be an evolving solution of (4.1) on a compact Riemannian manifold, let $\tilde{\lambda}$ be the least nonzero eigenvalue of the modified operator $-2\tilde{\Delta} + C\tilde{R}$, $C \geq \frac{1}{2}$ at time. i.e.,

$$\tilde{\lambda} = \inf \tilde{\mathcal{F}}_{BC} \text{ with } e^{-\tilde{f}} d\tilde{\mu}$$

then, we have

$$\frac{d\tilde{\lambda}}{dt} = 2\int_M |\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2 e^{-\tilde{f}} d\tilde{\mu} + 2(2C-1)\int_M |\tilde{R}_{ij}|^2 e^{-\tilde{f}} d\tilde{\mu} - \frac{2r}{n}\tilde{\lambda} \quad (4.9)$$

when r is nonpositive. If r is strictly negative, we have the following version of Theorem 10.

Theorem 16. *The least eigenvalue of* $-2\tilde{\Delta} + C\tilde{R}$ *is diffeomorphism invariance and non-decreasing under the normalized Ricci flow. The monotonicity is strict unless we are on the Einstein metric.*

Proof. (a) The first part of the Theorem is modelled after the first part of the proof of Theorem 10.

(b) The second part can be seen using equation (4.9)

$$\frac{d}{dt}\tilde{\lambda} \geq 0, \text{ where } r \leq 0.$$

(c) Examining (4.9), it is clear that it fails to classify the steady state of the least eigenvalue (as remarked in [11]), so we need a modified form of (4.9) to tell the class of Einstein metric involved, we however have

$$\frac{d\tilde{\lambda}}{dt} = 2\int_M \left|\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}} d\tilde{\mu} + 2(2C-1)\int_M \left|\tilde{R}_{ij} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}} d\tilde{\mu} - \frac{2r\tilde{\lambda}}{n}$$

$$+ \frac{4r}{n}\int_M \tilde{g}^{ij}(\tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} + \tilde{R}_{ij})e^{-\tilde{f}} d\tilde{\mu} - 2\int_M \left|\frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}} d\tilde{\mu}$$

$$+ 4(2C-1)\frac{r}{n}\int_M \tilde{g}^{ij}\tilde{R}_{ij}e^{-\tilde{f}} d\tilde{\mu} - 2(2C-1)\int_M \left|\frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}} d\tilde{\mu}$$

$$= 2\int_M \left|\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}} d\tilde{\mu} + 2(2C-1)\int_M \left|\tilde{R}_{ij} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}} d\tilde{\mu} - \frac{2r\tilde{\lambda}}{n}$$

$$+ \frac{4r}{n}\tilde{\mathcal{F}}_{\mathcal{B}C} - \frac{4Cr^2}{n}$$

$$= 2\int_M \left|\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}} d\tilde{\mu} + 2(2C-1)\int_M \left|\tilde{R}_{ij} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}} d\tilde{\mu} + \frac{2r}{n}(\tilde{\lambda} - 2Cr)$$

$$\geq 0$$

since by definition $\tilde{\lambda} \leq Cr$.

Corollary 17. *Under the normalized Ricci flow, the following monotonicity formula holds*

$$\frac{d\tilde{\lambda}}{dt} =$$

$$2\int_M \left|\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} + 2(2C-1)\int_M \left|\tilde{R}_{ij} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} \geq 0. \quad (4.10)$$

Equality is attained if and only if $\tilde{g}(t)$ *is Einstein and* $\tilde{f}$ *is a constant gradient function.*

Thus, we can rule out the existence of nontrivial expanding gradient Ricci breathers excepts those that are gradient solitons. If $C = \frac{1}{2}$ and $r \leq 0,$ we have the monotonicity formula

$$\frac{d\tilde{\lambda}}{dt} = 2\int_M \left|\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} + \frac{2r}{n}(\tilde{\lambda} - 2Cr) \geq 0 \qquad (4.11)$$

which simply implies that expanding breathers are necessarily expanding soliton.

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