Fundamental J. Modern Physics, Vol. 1, Issue 1, 2011, Pages 41-62
Published online: October 5, 2011

NONCLASSICAL SOLITARY WAVES FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATION

M. AGÜERO*, F. ONGAY and J. SÁNCHEZ-MONDRAGÓN

Facultad de Ciencias
Instituto Literario 100
Universidad Autónoma del Estado de México
Toluca 50000, Edo. México, México
e-mail: maaguerog@uaemex.mx

Abstract

We present an analysis of a generalized Φ^4 model for a chain of particles along a line subjected to nonlinear an-harmonic self-interaction potentials. Several analytical compact and peak like solutions are obtained considering appropriated restrictions on their speeds and jump conditions. We study their phase trajectories and show the stability conditions for each solution.

1. Introduction

The recent interest in the phenomenology of the nonlinear waves has sparked a resurgence of interests in the physics of classical kink-shaped $\tanh$ solutions, bell-shaped sech solutions and also nonclassical soliton solutions like compact bubble or kinks waves. These nonlinear wave phenomena are observed in uid dynamics, plasma, elastic media, optical bres, DNA, etc. It is well-known that traveling wave solutions of various Nonlinear Partial Differential Equations (NPDEs) play crucial roles in the study of nonlinear wave phenomena. In this fashion, the nonlinear Klein

Keywords and phrases: nonclassical solitary wave solutions, compactons, peakons, solitons, ODE, nonlinear differential equations.

*Corresponding author

Received April 18, 2011

Gordon equation has been extensively studied using different analytical and numerical approaches. Commonly, in those studies the model was analyzed by using several kinds of self interaction potentials. For Sine-Gordon like equation, we found systematic studies in Scolt et al. [1], Ablowitz et al. [2], Dodd et al. [3], and Ablowitz and Clarkson [4].

On the other hand after the introduction of an-harmonic terms for the substrate potential, suggested the emergence of special type of soliton structures called *compactons* and *peakons*. For instance, the compact like structures have the quality of absence of long tails that characterize classical soliton structures. In many aspects due to this property their interaction is similar to the interaction of hard spheres. As is suggested by its name, a compacton is a soliton structure with compact support. Compactons have originally been discovered as a special class of nonlinear waves in generalized versions of the KdV equation (Rosenau [5] and Rosenau and Hyman [6]).

For the case of discrete Klein-Gordon Equation the kink solutions can be obtained either by perturbation approaches [7] or by numerical techniques. If KG also includes an-harmonic interactions, then specific kink internal modes may be created [8]. In other various previous works [9-12], different solutions have been found to the nonlinear continuum version of Klein-Gordon equation with self an-harmonic interaction with and without harmonic substrate potential. In many cases, these approaches did not consider the stability conditions for each presented solution. In this paper, we study a general version of nonlinear Klein-Gordon equation than that presented in [11, 12]. In these papers was considered self an-harmonic interaction in addition to possessing a harmonic substrate potential. In order to study the model, we applied the mechanical analogy method which was able to demonstrate some peculiar properties of solutions. Some analytical solutions were obtained for this equation and it was drawing some similarities with the equation presented in [12] by a suitable choice of corresponding speed values.

In the first section, we will construct the general form of the nonlinear Klein-Gordon equation, with an-harmonic potential terms of substrate. We will provide the condition for continuously "gluing" different branches of solutions. Also, we will use the virial theorem and the second variation for obtaining stability conditions for solutions. Analytical solutions are presented under the trivial and condensed type of boundary conditions. We briefly describe the mechanical analogy method for each case. This approach will allow us to confirm or exclude the existence of available solutions. We evaluate the energy for each solution and the dispersion relation will be discussed by using virial relation and the second variation of the Hamiltonian. At the end, we present a short discussion on the existence of the solutions shown in [11, 12] as special ones of our equation.

2. The Equation of Motion

First, we construct the equation of motion for the system of identical particles along a chain with mass "*m*" subjected to substrate potential $V(\theta_n)$,

$$V(\theta_n) = \frac{\alpha\theta_n^2}{2} - \frac{\beta\theta_n^4}{4}, \tag{1}$$

that represents the interaction of the system with the background. The interaction between neighbor particles is being done by the potential $U(\theta_n - \theta_{n-1})$ that can be modeled by

$$U(\theta_n - \theta_{n-1}) = \frac{C_1}{2}(\theta_n - \theta_{n-1})^2 + \frac{C_2}{4}(\theta_n - \theta_{n-1})^4. \tag{2}$$

As we can notice, this equation is constructed taking in mind harmonic and an-harmonic potential parts. Here α, β, C_1 and C_2 are constants that control the barrier of double well potential and the forces of linear and nonlinear coupling, respectively. The equation of motion that describes a discrete set of *n* bodies under the interaction potential described by $V(\theta_n)$ and $U(\theta_n - \theta_{n-1})$ will take the following form

$$\begin{aligned} &\frac{d^2\theta}{dt^2} - C_1(\theta_{n+1} + \theta_{n-1} - 2\theta_n) \\ &+ C_2[(\theta_n - \theta_{n-1})^3 + (\theta_n - \theta_{n+1})^3] + \alpha\theta_n - \beta\theta_n^3 = 0. \end{aligned} \tag{3}$$

For the case when θ_n varies slowly, we can use the standard continuum approximation $\theta_n(t) \to \theta(x, t)$ and extend θ_{n+1} in terms of $\theta(x, t)$. Under these considerations making $x = \frac{\chi}{a}$ (where *a* is the distance between points of the lattice) and assuming that $3C_2\theta_x^2\theta_{xx} >> \left(\frac{C_1}{2}\right)\theta_{xxxx}$, from equation (3) we obtain

$$\theta_{tt} - C_1\theta_{xx} - 3C_2\theta_x^2\theta_{xx} + \alpha\theta - \beta\theta^3 = 0. \tag{4}$$

We will study the continuum version of the discrete nonlinear equation (3). Following the same approach done in [12], one can arrive at the following ordinary differential equation

$$\theta_s^4 - \frac{2(u^2 - C_1)}{3C_2}\theta_s^2 + \frac{1}{3C_2\beta}(\alpha - \beta\theta^2)^2 + \gamma = 0, \tag{5}$$

with γ being the constant of integration that can be considered equal to zero. By rewriting the parameters

$$a = \frac{2(u^2 - C_1)}{3C_2}, \qquad b = \frac{1}{3C_2\beta}, \tag{6}$$

we obtain the next equation

$$\theta_s^4 - a\theta_s^2 + b(\alpha - \beta\theta^2)^2 = 0. \tag{7}$$

That has the same structure of the similar equation discussed in the paper [12]. Nevertheless, it is possible to find new solutions as particular cases in the same fashion that was done in the mentioned paper [11].

3. Energy and Stability of Solutions

A. Jump conditions

During the process of finding solutions, we observed that these solutions contain several important branches. Next, for avoiding discontinuities, we need the proper point to "glue" the branches of the solutions and to build the physical acceptable solutions, so we evaluate the next equation

$$\int_{s_0-\varepsilon}^{s_0+\varepsilon}((u^2 - C_1)\theta_{ss} - 3C_2\theta_s^2\theta_{ss} + \alpha\theta - \beta\theta^3)ds = 0, \tag{8}$$

where s_0 is a critical point where the jump occurs and the first derivative supports discontinuities

$$\lim_{\varepsilon\to 0}\frac{\partial\theta}{\partial s}\Big|_{s_0-\varepsilon}^{s_0+\varepsilon} \neq 0. \tag{9}$$

After several calculations, we find the points at which we can make the jumping and it is given by the condition

$$\theta_s(s_0) = \pm\sqrt{\frac{u^2 - C_1}{3C_2}} = \pm\sqrt{\frac{a}{2}}. \tag{10}$$

B. Dispersion relations and energies

To obtain the terms of dispersion associated with each solution, we add as usual a small deviation (η) in the equation of motion, such that

$$\theta \to \theta_0 + \eta \tag{11}$$

with $|\eta| << 1$. We will consider the trivial boundary condition when $\theta_0 = 0$ and solutions on condensed state trivial boundary conditions $\theta_0 = 1$. Therefore, for the continuum version of equation (3), we find the linear equation that describes the small deviation vacuum states

$$\eta_{tt} - C_1\eta_{xx} + \alpha\eta = 0, \tag{12}$$

and the dispersion relation is

$$\omega^2 = C_1 k^2 + \alpha. \tag{13}$$

Therefore the solution is stable when

$$k^2 > -\frac{\alpha}{C_1}. \tag{14}$$

Since we restrict ourselves to the case of traveling waves, the Hamiltonian density can be rewritten as

$$H = \frac{u^2 + C_1}{2}\left(\frac{d\theta}{ds}\right)^2 + \frac{C_2}{2}\left(\frac{d\theta}{ds}\right)^4 + \frac{\alpha\theta^2}{2} - \frac{\beta\theta^4}{4}. \tag{15}$$

Consequently, the expression for the energy values can be written in terms of the variables θ_s and θ as

$$E = \int_{\theta_1}^{\theta_2}\left[\frac{u^2 + C_1}{2}\theta_s + \frac{C_2}{2}\theta_s^3 + \frac{\alpha\theta^2}{2\theta_s} - \frac{\beta\theta^4}{4\theta_s}\right]d\theta. \tag{16}$$

C. Virial relations

For better treatment, we reorganize the relations for energy values (16) in the following manner

$$H = \frac{u^2 + C_1}{2} E_1 + \frac{C_2}{2} E_2 + \frac{\alpha}{2} E_3 - \frac{\beta}{4} E_4 \tag{17}$$

with

$$E_1 = \int \theta_s d\theta, \quad E_2 = \int \theta_s^3 d\theta, \quad E_3 = \int \frac{\theta^2}{\theta_s} d\theta, \quad E_4 = \int \frac{\theta^4}{\theta_s} d\theta, \tag{18}$$

and by considering the transformation $\theta_a = \theta(as)$ in the total Hamiltonian, we have

$$H[\theta_a] = a \frac{u^2 + C_1}{2} E_1 + a^3 \frac{C_2}{2} E_2 + \frac{1}{a}\left(\frac{\alpha}{2} E_3 - \frac{\beta}{4} E_4\right). \tag{19}$$

Evaluating $(dH/da)|_{a=1}$, we can obtain the virial relation

$$\frac{u^2 + C_1}{2} E_1 + 3\frac{C_2}{2} E_2 - \frac{\alpha}{2} E_3 + \frac{\beta}{4} E_4 = 0, \tag{20}$$

from which we can find that

$$\frac{u^2 + C_1}{2} E_1 = -3\frac{C_2}{2} E_2 + \frac{\alpha}{2} E_3 - \frac{\beta}{4} E_4. \tag{21}$$

Then, if we substitute this condition into the equation for energy, we have

$$H = \alpha E_3 - C_2 E_2 - \frac{\beta}{2} E_4. \tag{22}$$

Next, the second variation of the Hamiltonian $H[\theta_a]$ takes the form

$$(d^2 H/da^2)|_{a=1} = 3C_2 E_2 + \alpha E_3 - \frac{\beta}{2} E_4 > 0. \tag{23}$$

The conditions of stability by using the second variation $\delta^2 H$ and the energies should be analyzed for each particular case.

Let us analyze some particular cases. The first one we consider when $u = \sqrt{C_1}$ (i.e., $u^2 - C_1 = 0$), the second case will take place when $C_2 = 0$ (i.e., when the potential interaction between neighboring particles is of harmonic type). Finally, in the third case we will develop the conditions for which is obtained a solution to equation (7). It should be noted that the particular case for $C_2 = 0$ was analyzed in the work [12]. Complementary to the stability analysis for each structure, it will take place the study of their phase trajectories. Starting from equation (4) the subsequent algebra yields the following expression

$$\theta_{ss} = \frac{-\alpha\theta + \beta\theta^3}{(u^2 - C_1) - 3C_2\theta_s{}^2} \tag{24}$$

which will help us to determine the direction of the phase trajectories.

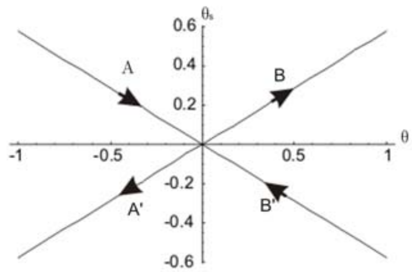

Figure 1. Phase trajectories when $\beta = -1$ and $C_2 = 3$.

4. Solutions with Trivial Boundary Conditions

As usual, these solutions are being restricted by the following conditions

$$s \to \pm\infty, \quad \theta_s \to 0, \quad \theta \to 0. \tag{25}$$

Applying these conditions to equation (7) yields $\alpha = 0$. Therefore we have several important cases

A. Case I. $u^2 - C_1 = 0$

For the case when the velocity of traveling waves satisfies the relation $u^2 - C_1 = 0$, equation (3) is reduced to the next one

$$\theta_s^4 + b\beta^2\theta^4 = 0. \tag{26}$$

From here, the phase trajectories for equation (7) are given by the function $\theta_s(\theta)$,

$$\theta_s = \pm\left(\frac{-\beta}{3C_2}\right)^{1/4}\theta. \tag{27}$$

Thus, the phase trajectories that correspond to the solutions (26) will be the straight lines drawn in Figure 1.

By solving equation (26), we get

$$\theta(s) = Ae^{\pm\left(\frac{-\beta}{3C_2}\right)^{1/4} s}, \tag{28}$$

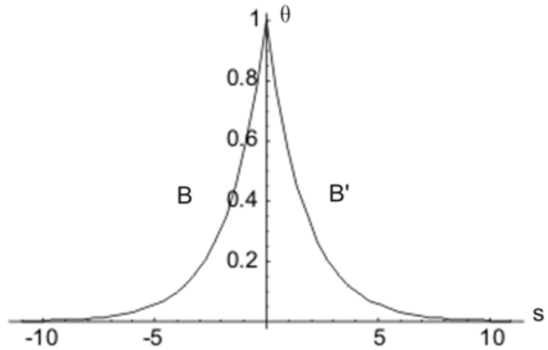

Figure 2. Peak $\beta = 1$ and $C_2 = 3$.

whose figure corresponds to the “peak”, see Figure 2.

To construct the solution, we note that the point at which we glue the branches of the function is given by the condition (10). This condition means that is possible to build appropriated solutions due to the relation $\theta_s = 0.$ In the similar way, the peak obtained in [11] has the similar behavior in its phase trajectories.

B. Case II. $C_2 = 0$

For the case when the inter-particle interaction between particles is an-harmonic type, i.e., for $C_2 = 0,$ equation (24) transforms to

$$\theta_s^2 - \frac{1}{2\beta(u^2 - C_1)}(\alpha - \beta\theta^2)^2 = 0. \tag{29}$$

By applying the trivial boundary condition when $\alpha = 0$ and using equation (5) it is possible to obtain the function $\theta_s(\theta)$,

$$\theta_s = \sqrt{\frac{\beta}{2(u^2 - C_1)}}\theta^2, \tag{30}$$

from which we can get the phase trajectories that are depicted in Figure 3.

Solving equation (30) the branches of the analytical solutions appear

$$\theta(s) = \frac{1}{\pm \frac{\beta}{\sqrt{2\beta(u^2 - C_1)}} s + C}. \tag{31}$$

One of the available profiles corresponds to a similar Delta function Dirac, see Figure 4.

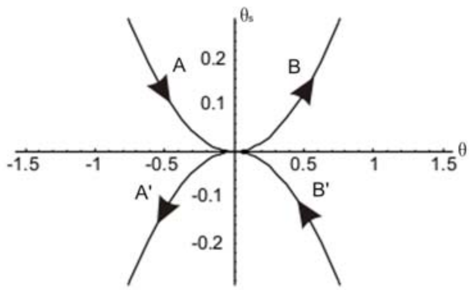

Figure 3. Phase trajectories for $\beta = 1$ and $u^2 - C_1 = 2$.

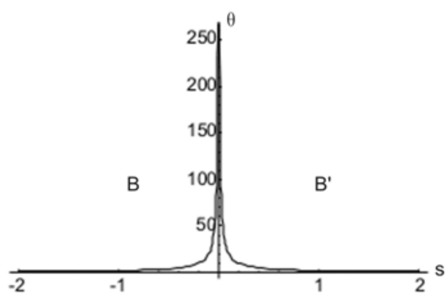

Figure 4. Profile of the solution when $\beta = 1$ and $u^2 - C_1 = 2$.

C. Case III

From equation (7) by using the trivial boundary conditions $(\alpha = 0)$, we obtain the following equation

$$\theta_s^4 - a\theta_s^2 + b\beta^2\theta^4 = 0. \tag{32}$$

Thus, we can find the function $\theta_s(\theta)$,

$$\theta_s = \sqrt{\frac{a}{2}}\sqrt{1 \pm \sqrt{1 - \frac{3C_2\beta}{(u^2 - C_1)^2}\theta^4}}. \tag{33}$$

In the case of negative sign of the previous equation, we make $\theta \to 0$ and obtain $\theta_s \to 0.$ The corresponding picture of the phase trajectories is presented in Figure 5, and its corresponding solution is shown in Figure 6.

The analytical solution of this equation can be found by utilizing $\sigma = \dfrac{3C_2\beta}{(u^2 - C_1)^2}$ and similarly

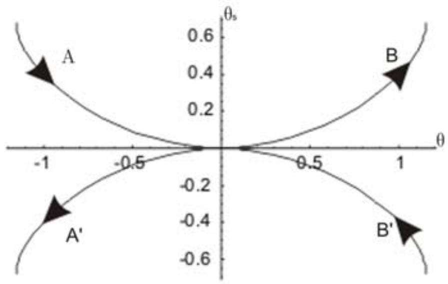

Figure 5. Phase trajectory when $\beta = 1,\ C_2 = 3$ and $u^2 - C_1 = 2.$

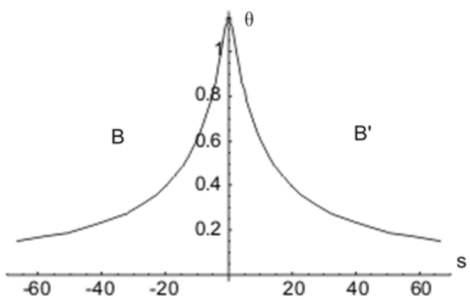

Figure 6. A peak solution when $\beta = 1,\ C_2 = 3,\ \sigma = 1.1547$ and $u^2 - C_1 = 2.$

as done in [12], we write the transcendental equation for the solution

$$\pm\sqrt{\frac{a\sqrt{\sigma}}{2}}(s - s_0) = -\mathrm{Arctan}\left[-1 + \sqrt{\frac{2\sqrt{\sigma}\theta^2}{1 + \sqrt{1 - \sigma\theta^4}}}\right] - \mathrm{Arctan}\left[1 + \sqrt{\frac{2\sqrt{\sigma}\theta^2}{1 + \sqrt{1 - \sigma\theta^4}}}\right]$$

$$-\frac{1}{2}\,\mathrm{Ln}\left[-1 + \sqrt{\frac{2\sqrt{\sigma}\theta^2}{1 + \sqrt{1 - \sigma\theta^4}}} - \frac{\sqrt{\sigma}\theta^2}{1 + \sqrt{1 - \sigma\theta^4}}\right]$$

$$-\frac{1}{2}\,\mathrm{Ln}\left[1+\sqrt{\frac{2\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}}-\frac{\sqrt{\sigma}\theta^2}{1+\sqrt{1-\sigma\theta^4}}\right]$$

$$-4\sqrt{\frac{1+\sqrt{1-\sigma\theta^4}}{2\sqrt{\sigma}\theta^2}}. \tag{34}$$

The jump condition (10) appears when $\theta_s = \pm\sqrt{\frac{a}{2}}$, i.e., for $\theta = \pm\sqrt[4]{\frac{a}{2\sqrt{b}\beta}} = \pm\sqrt[4]{\frac{1}{\sigma}}$. This condition will give us the height of the peak solution and for given equation (13) when $\alpha = 0$, we have the dispersion relation $\omega^2 = C_1 k^2$. Thus the stability condition is satisfied when $k^2 > 0$. The compact solution for the trivial boundary condition (25) in [12] is acceptable since it corresponds to the positive sign of equation (33).

1. Energies

For the obtained solutions of Case III, we can evaluate their energies by using equations (18), (22), $\alpha = 0$ and also the next equation

$$\theta_s = \sqrt{\frac{a}{2}}\sqrt{1-\sqrt{1-\sigma\theta^4}}. \tag{35}$$

For the energy expressions after a long algebra, we have

$$E = \frac{a^{1/2}(u^2 - C_1)}{8\sigma^{1/4}}\left(\frac{7}{6}\sqrt{2} - 4 - 3n\pi - \frac{5}{2}\,\mathrm{Log}\left[1+\sqrt{2}\right] - \frac{7(4m+3)\pi}{2}\right) \tag{36}$$

with m, n integer numbers. Now by using the value of the energy portion E_1 from equation (18)

$$E_1 = \frac{\sqrt{a}}{2\sigma^{1/4}}\left(\sqrt{2} - n\pi - \frac{4m+3}{2}\,\pi\right) \tag{37}$$

and its subsequent substitution in the virial relation (20), we can evaluate the suitable velocity values for the traveling solutions with trivial boundary condition

$$u = \sqrt{\frac{24 + 9\sqrt{2} - 26n\pi + 17\mathrm{Log}[1+\sqrt{2}] + (4m+3)\pi}{24 + \sqrt{2} - 10n\pi + 17\mathrm{Log}[1+\sqrt{2}] + 9(4m+3)\pi}C_1}, \tag{38}$$

that is related directly to the parameter C_1. Also we can obtain the condition of stability by using the values of the energies taking in mind the second variation of the Hamiltonian (23),

$$\delta^2 H = \frac{a^{1/2}(u^2 - C_1)}{8\sigma^{1/4}}\left(12 - \sqrt{2} - 18n\pi + 9\text{Log}[1+\sqrt{2}] + \frac{9(4m+3)\pi}{2}\right)$$
$$-\frac{a^{1/2}(u^2 - C_1)}{16\sigma^{1/4}}\left(3\sqrt{2} + 18n\pi - \text{Log}[1+\sqrt{2}] + \frac{(4m+3)\pi}{2}\right) > 0. \tag{39}$$

From here it is easy to check the values of n and m are restricted by the following expression

$$m > \frac{108n\pi + 2\sqrt{2} - 99 - 38\text{Log}[1+\sqrt{2}]}{68}. \tag{40}$$

5. Nontrivial Boundary Conditions

Next, we consider the condensate type of boundary conditions

$$s \to \pm\infty, \quad \theta_s \to 0, \quad \theta \to \theta_0 = \text{const.} \tag{41}$$

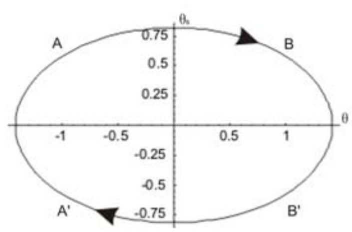

Figure 7. Phase trajectories when the parameter values are $\alpha = 2$, $\beta = 1$ and $C_2 = -3$.

We apply it to equation (7) and obtain

$$\theta_0 = \pm\sqrt{\frac{\alpha}{\beta}}. \tag{42}$$

Now, we analyze the possible solution of the system

A. Case I. $u^2 - C_1 = 0$

For the case when $u^2 - C_1 = 0$, from (3) we have the next equation

$$\theta_s^4 + b(\alpha - \beta\theta^2)^2 = 0, \tag{43}$$

and the jump condition is given by

$$\theta_s = \pm(-b)^{1/4}(\alpha - \beta\theta^2)^{1/2} = 0, \tag{44}$$

that gives the restrictions $C_2 < 0$, $b < 0$ when we try to construct the phase trajectories. When solving this equation we obtain the next branches

$$\theta(s) = \pm\sqrt{\frac{\alpha}{\beta}}\sin(\pm(-b\beta^2)^{1/4}s) \tag{45}$$

with a proper choice of the branches A and B of the phase trajectory, the solution that can be built is a typical compact kink represented in Figure 8. Taking the branches A, B, B′ and A′ of the phase trajectories, we can build also a compact pulse of the bell form, see Figure 9. The other structure that we can construct could be a periodic solution like sinus function, Figure 10.

Thus, the compact kink and compact pulse with velocity $u = \sqrt{C_1}$ will have a dispersion relation given by expression (13)

$$\omega^2 = u^2k^2 + \alpha, \tag{46}$$

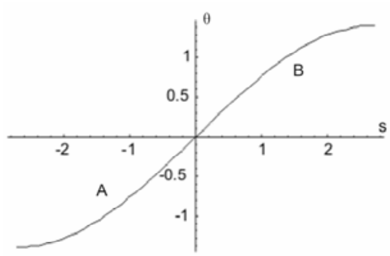

Figure 8. Compact kink for $\alpha = 2$, $\beta = 1$ and $C_2 = -3$.

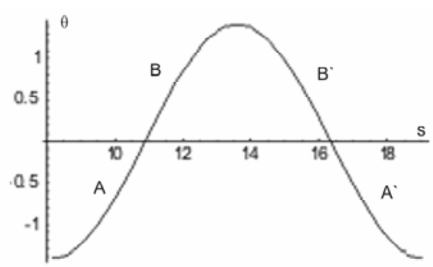

Figure 9. Compact pulse for $\alpha = 2$, $\beta = 1$ and $C_2 = -3$.

and the solution is stable when

$$k^2 > \frac{-\alpha}{C_2}. \tag{47}$$

The solution will be supported at the condensate for $\theta_0 = \pm\sqrt{\frac{\alpha}{\beta}}$. By replacing it in the equations for energies given in the first section when $u^2 = C_1$, we get

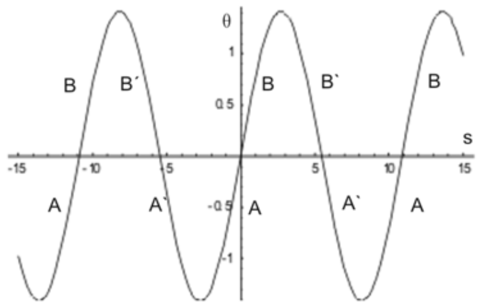

Figure 10. Periodical solution when $\alpha = 2$, $\beta = 1$ and $C_2 = -3$.

$$E_1 = (-b\beta^2)^{\frac{1}{2}}\left(\frac{s}{2} + \frac{\sin[2(-b\beta^2)^{\frac{1}{4}}s]}{4(-b\beta^2)^{\frac{1}{4}}}\right),$$

$$E_2 = \frac{1}{32}(-b\beta^2)^{\frac{3}{4}}(12(-b\beta^2)^{\frac{1}{4}}s + 8\sin[2(-b\beta^2)^{\frac{1}{4}}s] + \sin[4(-b\beta^2)^{\frac{1}{4}}s]),$$

$$E_3 = \frac{s}{2} - \frac{\sin[2(-b\beta^2)^{\frac{1}{4}}s]}{4(-b\beta^2)^{\frac{1}{4}}},$$

$$E_4 = \frac{12(-b\beta^2)^{\frac{1}{4}}s - 8\sin[2(-b\beta^2)^{\frac{1}{4}}s] + \sin[4(-b\beta^2)^{\frac{1}{4}}s]}{32(-b\beta^2)^{\frac{1}{4}}}. \tag{48}$$

Since the condensate value is given by $\theta = \pm\theta_0$, the energies will be evaluated between $s_0 = \pm\frac{(4n+1)\pi}{2(-b\beta^2)^{1/4}}$ for $-\theta_0$ and $s_1 = \pm\frac{(4m+3)\pi}{2(-b\beta^2)^{1/4}}$ for θ_0. Therefore, the energy for the compact kink E_{ck} that is contained in the trajectories A and B is calculated as

$$E_{ck} = \frac{\alpha(2(m-n)+1)\pi}{2(-b\beta^2)^{\frac{1}{4}}} - C_2 \frac{3(2(m-n)+1)\pi}{8}(-b\beta^2)$$

$$- \frac{3\beta(2(m-n)+1)\pi}{16(-b\beta^2)^{\frac{1}{4}}}. \tag{49}$$

Using the virial relation (20),

$$\frac{u^2 + C_1}{2} E_1 + \frac{3C_2E_2}{2} - \frac{\alpha}{2} E_3 + \frac{\beta}{4} E_4 = 0 \tag{50}$$

and $u^2 = C_1,$ the velocity values are strait-fully obtained

$$u = \sqrt{\frac{8\alpha - 3\beta}{16(-b\beta^2)^{\frac{1}{4}}} + \frac{3\beta}{4}}. \tag{51}$$

From the second variation for the compact kink, we have

$$\frac{9C_2}{8}(-b\beta^2)^{\frac{5}{4}} + \frac{\alpha}{2} > \frac{3\beta}{16}. \tag{52}$$

The other solution the compact pulse that is determined in the trajectories $\mathrm{A}, \mathrm{B}, \mathrm{B}'$ and $\mathrm{A}',$ shows the following value of energy E_{cp}:

$$E_{cp} = \frac{\alpha q\pi}{(-b\beta^2)^{\frac{1}{4}}} - \frac{3C_2q\pi}{2}(-b\beta^2) - \frac{3\beta q\pi}{8(-b\beta^2)^{\frac{1}{4}}}, \tag{53}$$

$$q = m + n' - m' - n, \tag{54}$$

m, n, m', n' being integer numbers. From the second variation for the compact pulse, we have

$$\frac{9C_2(-b\beta^2)^{\frac{5}{4}}}{2} + \alpha > \frac{3\beta}{8}. \tag{55}$$

B. Case II. $C_2 = 0$

For this case, we have the next expression of ODE

$$\theta_s^2 - \frac{1}{2\beta(u^2 - C_1)}(\alpha - \beta\theta^2)^2 = 0, \tag{56}$$

and the phase trajectories are obtained by the next equation

$$\theta_s = \pm\frac{1}{\sqrt{2\beta(u^2 - C_1)}}(\alpha - \beta\theta^2). \tag{57}$$

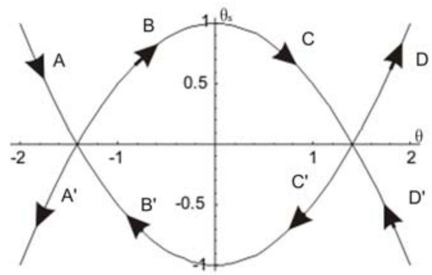

Figure 11. Phase trajectory for $\alpha = 2$, $\beta = 1$ and $u^2 - C_1 = 2$.

From here we can obtain two solutions, the first one constructed with the branches B and C, and the second solution with C' and C, as we can observe from Figure 11. Solving equation (57), we obtain

$$\theta(s) = \pm\sqrt{\frac{\alpha}{\beta}}\,\mathrm{Tanh}\sqrt{\frac{\alpha}{2(u^2 - C_1)}}\,s. \tag{58}$$

From the analysis of phase trajectories and with the proper election of the branches, we can obtain two different solutions that are depicted in Figure 12. We have here a kink that is constructed with the branches B and C. Figure 13 represents the gray soliton constructed with the aid of the C' and C branches.

The trajectories A, A', D and D' correspond to solutions outside the condensate. However these solutions could not be built because there is no possibility to glue the branches.

The kink solution is emerging in the condensate defined by $\theta = \pm\sqrt{\frac{\alpha}{\beta}}$, their energy values can be obtained from

$$E_1 = 2\alpha\sqrt{\frac{\alpha}{\beta}} - 2\frac{\beta\left(\frac{\alpha}{\beta}\right)^{3/2}}{3},$$

$$E_3 = -2\frac{1}{\beta}\sqrt{\frac{\alpha}{\beta}} + \frac{2\sqrt{\alpha}\operatorname{arctanh}(1)}{\beta^{\frac{3}{2}}}.$$

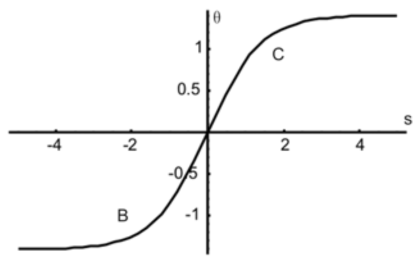

Figure 12. Kink like solution when $\alpha = 2$, $\beta = 1$ and $u^2 - C_1 = 2$.

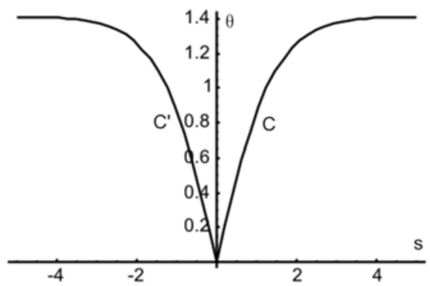

Figure 13. Gray soliton for when $\alpha = 2$, $\beta = 1$ and $u^2 - C_1 = 2$.

$$E_4 = -\frac{\alpha}{\beta^2}\sqrt{\frac{\alpha}{\beta}} - \frac{2}{3\beta}\left(\frac{\alpha}{\beta}\right)^{3/2} + \frac{\alpha^{3/2}\operatorname{arctan h}(1)}{\beta^{5/2}}. \tag{59}$$

C. Case III

From the relation (7) by using the condensate boundary conditions, one can obtain

$$\theta_s^4 - \alpha\theta_s^2 + b(\alpha - \beta\theta^2)^2 = 0, \tag{60}$$

by utilizing the same approach of the paper [12], we have the phase trajectory determined by the equation

$$\theta_s = \sqrt{\frac{a}{2}}\sqrt{1 \pm \sqrt{1 - (1 - f\theta^2)^2}}, \tag{61}$$

that is depicted in Figure 14.

Analyzing the phase trajectories (14), we can select only the paths denoted by dotted lines that represent a discontinuity, the possible trajectories will be those of Figure 15.

This trajectory corresponds to the function (61) with negative sign. The analytical solution of equation (60) can be obtained by using parameter restriction α,

$$\alpha = \sqrt{\frac{\beta(u^2 - C_1)^2}{3C_2}}, \tag{62}$$

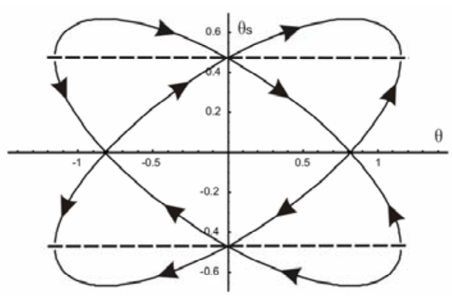

Figure 14. Phase of trajectory for $\alpha = 2/3$, $\beta = 1$, $u^2 - C_1 = 2$ and $C_2 = 3$.

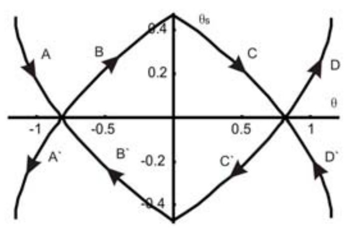

Figure 15. Phase trajectory para $\alpha = 2/3$, $\beta = 1$, $u^2 - C_1 = 2$ and $C_2 = 3$.

being $\sigma\alpha = 1$. From equation (61) with negative sign, we obtain

$$\pm 2\sqrt{fa}(s - s_0) = -\mathrm{Arcsin}(1 - f\theta^2) - \mathrm{Ln}\left(\frac{1 + \theta\sqrt{2f - f^2\theta^2}}{2}\right) \tag{63}$$

with

$$f = \frac{\beta}{\alpha}, \quad a = \frac{2(u^2 - C_1)}{3C_2}. \tag{64}$$

This transcendental equation possesses two branches that allow us to build the solutions. Using the phase trajectories, the available solutions that we can find for the function (61) are kink like structures (17) for the branches B and C. The anti-bubble solution of bell form is constructed using the branches B and B' and the peak like solution is built by the branches D and D'.

The solutions (63) satisfy the jump conditions (10), i.e., if we substitute in equation (61), it is observed that the points in which we can glue the appropriated branches for $\theta = 0, \pm\sqrt{\frac{2}{f}}$, are precisely the point where we glue the solutions for making the anti-bubble

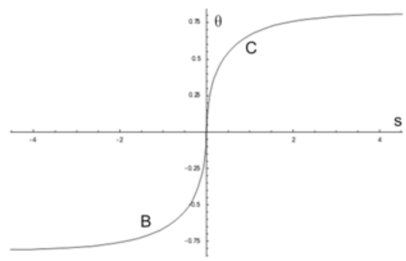

Figure 16. Kink para $\alpha = 2/3$, $\beta = 1$, $u^2 - C_1 = 2$ and $C_2 = 3$.

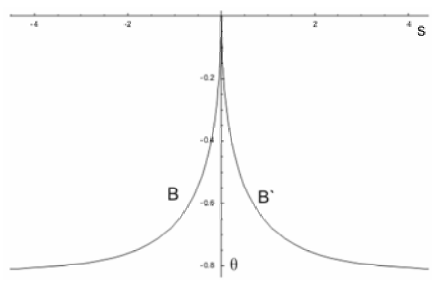

Figure 17. Anti-bubble solution when $\alpha = 2/3$, $\beta = 1$, $u^2 - C_1 = 2$ and $C_2 = 3$.

structure. The velocity of these solutions is defined by

$$u = \sqrt{\frac{3C_2\alpha}{\beta} + C_1}, \tag{65}$$

and the dispersion relation is given by (13). It is worthy to mention that the kink solution obtained in the work [9] also can be derived by our equation.

For the solution obtained, we substitute the velocity for the nontrivial solution and by using the next expression

$$\theta_s = \sqrt{\frac{a}{2}}\sqrt{1-\sqrt{1-(1-f\theta^2)^2}}, \tag{66}$$

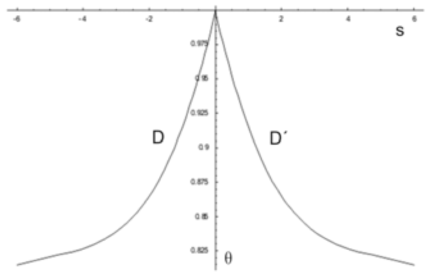

Figure 18. Peak like solution outside of condensate for $\alpha = 2/3$, $\beta = 1$, $u^2 - C_1 = 2$ and $C_2 = 3$.

we can get the available energy values

$$E = \gamma\left(\frac{u^2 + C_1}{2}E_1 + \frac{C_2}{2}E_2 + \frac{\alpha}{2}E_3 - \frac{\beta}{4}E_4\right),$$

$$\gamma = \frac{\sqrt{f}\,\theta}{\sqrt{2}\left(\sqrt{1+\sqrt{1-(1-f\theta^2)^2}} - \sqrt{1-\sqrt{1-(1-f\theta^2)^2}}\right)} \tag{67}$$

with the following expressions for different parts of the energy equation

$$E_1' = \frac{1}{2}\sqrt{\frac{a}{f}}\left[(1-f\theta^2) - \operatorname{Arcsin}(1-f\theta^2) + \sqrt{1-(1-f\theta^2)^2}\right],$$

$$E_2' = \frac{1}{16}\sqrt{\frac{a^3}{f}}\Big[8(1-f\theta^2) - 6\operatorname{Arcsin}(1-f\theta^2) + 4\sqrt{1-(1-f\theta^2)^2}$$

$$- 2(1-f\theta^2)\sqrt{1-(1-f\theta^2)^2} - \sqrt{1-4(1-f\theta^2)^2(1-(1-f\theta^2)^2)}\Big],$$

$$E_3' = \frac{1}{\sqrt{af^3}}\left[(1-f\theta^2) - \sqrt{1-(1-f\theta^2)^2} - \log\frac{1-\sqrt{1-(1-f\theta^2)^2}}{2}\right],$$

$$E_4' = \frac{1}{4\sqrt{af^5}}[8(1-f\theta^2) - 4\sqrt{1-(1-f\theta^2)} + 2(1-f\theta^2)\sqrt{1-(1-f\theta^2)}$$

$$+2\arcsin(1-f\theta^2)-4\log\frac{1-\sqrt{1-(1-f\theta^2)^2}}{2}$$

$$+\sqrt{1-4(1-f\theta^2)^2(1-(1-f\theta^2)^2)}]. \tag{68}$$

Each solution has the energy determined by its phase trajectories and has to be validated in the proper limit $\theta = 0$, $\theta_0 = \pm\sqrt{\frac{\alpha}{\beta}} = \pm\sqrt{\frac{1}{f}}$ y $\theta = \pm\sqrt{\frac{2}{f}}$. For example, the energy for the kink is

$$E = \frac{u^2 - C_1}{\sqrt{2^5\sqrt{\beta(3C_2)^3}}}(n - n')\pi, \tag{69}$$

the virial relation gives the next condition

$$u^2 = \frac{33}{41}C_1. \tag{70}$$

Calculating the energies, we found that the energies for the anti-bubble and peak like solutions correspondingly take divergent values.

6. Conclusions

We studied a system of identical particles subjected to nonlinear forces along a line and obtained nonlinear equation that in some sense is more general than that presented in works [9-12]. The jump condition method is used to construct wide classes of nonclassic traveling wave solutions of the NPDEs arising in nonlinear physics, such as Klein-Gordon equation. This nonlinear equation supports different analytical nonclassical solutions compacton bubble, kink, and peakons. These solutions were obtained making suitable speed restrictions in the space of parameter values. We ensure that solutions are properly constructed by given their phase trajectories and the fulfillment of gluing conditions. Additionally, we found dispersion relations and their energies that determine the stability and velocities of obtained solutions.

Acknowledgement

MAAG would like to express sincerely thanks to the SIEA UAEM for supporting him to spend a research stay at the Institute of General Physics, Russian Academy of Science, where this work was completed.

References

[1] A. C. Scolt, F. F. Chu and D. W. Mclaughlin, Proc. IEEE 61 (1973), 1443.

[2] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segar, Phys. Rev. Lett. 30 (1973), 1262.

[3] R. K. Dodd et al., Solitons and Nonlinear Wave Equations, Academic Press Inc. Ltd., London, 1982.

[4] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, London, 1991.

[5] P. Rosenau, Phys. Rev. Lett. 73 (1994), 1737-1740.

[6] P. Rosenau and J. M. Hyman, Phys. Rev. Lett. 70 (1993), 564-567.

[7] S. Flach and K. Kladko, Phys. Rev. E 54 (1996), 2912.

[8] F. Zhang, Phys. Rev. E 54 (1996), 4325.

[9] S. Dusel, P. Michaux and M. Remoissenet, Phys. Rev. E 57 (1998), 2320.

[10] T. Dinda and M. Remoissenet, Phys. Rev. E 60 (1999), 6218.

[11] M. Agüero and M. Paulin, Phys. Rev. E 63 (2001), 046606.

[12] M. Agüero, M. L. Najera, J. Aguilar and J. Sánchez, Int. J. Theor. Phys. 48 (2009), 2098.