Fundamental Journal of Mathematics and Mathematical Sciences
Vol. 7, Issue 1, 2017, Pages 33-50
This paper is available online at http://www.frdint.com/
Published online February 20, 2017

LORENTZIAN *n*-ELLIPSES

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Abstract

In this paper, we study the Lorentzian 3 and 4-ellipses in the Lorentzian plane. We give a geometric process to construct pure Lorentzian n-ellipses, $n \geq 3$, with foci the vertices of pure polygons in the Lorentzian plane. Finally, we show some families of them with $n \geq 1$.

1. Introduction

In the third century B. C., Apollonius characterized the points of the conics by their distances from two lines. By considering a conic as a section of a circular cone, he deduced important geometric properties of this characterization, which is equivalent to the current equation of a conic, [3]. In particular, an ellipse may be

Keywords and phrases: Lorentzian n-ellipse, pure curves, pure polygons, Lorentzian plane.

2010 Mathematics Subject Classification: 53B20, 53C50.

This work is supported by the Universidad Nacional del Sur (Grant no. PGI 24/ZL12).

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Received November 22, 2016; Accepted December 7, 2016

defined by the set of points P in the Euclidean plane such that the sum of distances of them to two focal points F_1 and F_2 is constant.

Standard definition of curves with constant distance sum to three focal points or more than three focal points are well known even from the seventeenth century (Descartes, 1638; Maxwell, 1847), [8].

Applications of these curves to different problems of nuclear sciences, engineering and theory of optimization are very contemporary even today, [6, 8]. For example, the curves with three focal points are used through analysis of one specific solution in urban and spatial planning was presented in [8].

In Lorentzian plane, the ellipses have been studied by Hano and Nomizu, [5], among others.

In this paper, in Section 3, we introduce the Lorentzian n-ellipses as curves with constant distance sum to n fixed points, called focal points, on surfaces in the Lorentzian 3-space.

Our purpose is to show families of pure timelike curves in the Lorentzian plane which are Lorentzian n-ellipses with the vertices of a pure spacelike polygon as focal points.

In order to do it, we will give the equations of Lorentzian 1 and 2-ellipses in the Lorentzian plane.

In Section 4, we will construct a Lorentzian 3-ellipse which is a pure timelike curve and whose focal points are vertices of a pure spacelike triangle. In addition, we will show a relationship between this Lorentzian 3-ellipse and a certain curve on the null cone in Lorentzian 3-space.

In Section 5, we will construct a Lorentzian 4-ellipse.

In Section 6, we will probe the main result of this work:

Theorem 1. *Given* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m$, *in the Lorentzian plane there exist families* $\{g_n\}_{n \in \{1, \ldots, m\}}$ *of pure timelike curves* (*resp.*, *pure spacelike curves*) *and* $F_1, \ldots, F_m$ *points such that for all* $n \in \{1, \ldots, m\}$, g_n *is a Lorentzian n-ellipse with* $F_1, \ldots, F_n$ *vertices of a pure spacelike polygon* (*resp.*, *pure timelike polygon*) *as foci*, *and*

$$\sum_{i=1}^{n} d(F_i,\ P) = \lambda_n,$$

for all $P \in g_n$.

In Section 2, preliminaries, we will remind basic notions in Lorentzian geometry about our concerns.

2. Preliminaries

Let x and y be two vectors in the n-dimensional vector space $\mathbb{R}^n$, $n \in \{2; 3\}$. As it is well known, [2, 7], the Lorentzian inner product of x and y is defined by

$$\langle x,\ y\rangle = \sum_{i=1}^{n-1} x_i y_i - x_n y_n.$$

Thus, the square ds^2 of an element of arc-length is given by

$$ds^2 = \sum_{i=1}^{n-1} dx_i^2 - dx_n^2.$$

The space $\mathbb{R}^n$ furnished with this metric is called a Lorentz n-space or n-dimensional Lorentzian space. We write L^n instead of $(\mathbb{R}^n,\ ds)$, $n \in \{2; 3\}$.

We say that a vector $x \in L^n$ is timelike if $\langle x,\ x\rangle < 0$, spacelike if $\langle x,\ x\rangle > 0$ and null if $\langle x,\ x\rangle = 0$. The null vectors also said to be lightlike. The *norm* $\|x\|$ is defined to be $\sqrt{|\langle x,\ x\rangle|}$. The distance between two points $P,\ Q \in L^n$ is defined to be $d(P,\ Q) = \|\overrightarrow{PQ}\|$.

In L^n, x is orthogonal to y if $\langle x,\ y\rangle = 0$, $x \neq y \neq 0$.

In L^2, let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. A timelike vector x is future-pointing (resp., past-pointing) if $\langle x,\ e_2\rangle < 0$ (resp., $\langle x,\ e_2\rangle > 0$). A spacelike vector x is future-pointing (resp., past-pointing) if $\langle x, e_1 \rangle > 0$ (resp., $\langle x, e_1 \rangle < 0$).

Let x and y be two future-pointing spacelike vectors in L^2. We say that $\alpha, \alpha \geq 0,$ is the (oriented) angle from x to y if $\begin{pmatrix} ch(\alpha) & sh(\alpha) \\ sh(\alpha) & ch(\alpha) \end{pmatrix} \cdot x = y.$ Hence, $ch(\alpha) = \frac{\langle x, y \rangle}{\|x\| . \|y\|}$, (cf. [2]).

We shall give a surface $M \in L^3$ by expressing its coordinates x_i as functions of two parameters in certain interval (see [7] for more details). We consider the functions x_i to be real functions of real variables.

We say that a differentiable map $\alpha : I \to M$, where $I \subset \mathbb{R}$ is an open interval, is a differentiable curve in *M*. This curve is called pure timelike, pure spacelike or pure lightlike curve if at every point, its tangent vector is timelike, spacelike or lightlike, respectively.

In classical way, $T[A, B, C]$ denotes the triangle with vertices A, B, C which are non collinear points in Lorentzian plane. According to Birman and Nomizu, [2], by a pure spacelike triangle we mean a triangle $T[A, B, C]$ such that $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{AC}$ are spacelike vectors. We will call the middle vertex to the vertex *B* such that the angle $\hat{B}$ between $\overline{AB}$ and $\overline{BC}$ looks more like the exterior angle to Euclid. That is, the vertex *B* is a middle vertex of $T[A, B, C]$ if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same spacelike orientation.

In [4], for each $m \geq 1$, it is defined a polygonal path of order *m* as a set of points of the form $[P_0, P_1, ..., P_{m+1}] = \overline{P_0P_1} \cup \overline{P_1P_2} \cup \cdots \cup \overline{P_mP_{m+1}}$, with the points $P_0, P_1, ..., P_{m+1} \in L^2$ as vertices and the named segments as sides. If the polygonal path $[P_0, P_1, ..., P_{m+1}]$ is closed and if no three of its vertices lie on a line, then it is called a polygon and it is denoted $\mathbf{P}[P_0, P_1, ..., P_m]$.

A polygon $\mathbf{P} = \mathbf{P}[P_0, P_1, ..., P_m]$ is said to be pure spacelike, pure timelike or pure lightlike if every side of $\mathbf{P}$ is spacelike, timelike or lightlike, respectively.

Let us recall that the polygon with at most two different vertices is usually called a degenerated polygon. In what follows, will consider non degenerated polygons.

In [2], the following two results are proved:

1. *Reversed triangular inequality*. Let x and y be future-pointing timelike vectors in L^2. Then, $\|x + y\| \geq \|x\| + \|y\|$.

2. *Hyperbolic cosine law*. Let $T[A, B, C]$ be a pure triangle. Then

$$a^2 = b^2 + c^2 - 2bc\ ch(\hat{A}),$$

$$c^2 = a^2 + b^2 - 2ab\ ch(\hat{C}),$$

$$b^2 = a^2 + c^2 + 2ac\ ch(\hat{B}),$$

where $a = \|\overrightarrow{BC}\|, b = \|\overrightarrow{AC}\|, c = \|\overrightarrow{AB}\|$ and $\hat{A}, \hat{B}, \hat{C}$ denote the angles at $A, B, C,$ respectively.

3. Lorentzian 1 and 2-ellipses

Now, we introduce a natural class of generalized multi foci curve from the Apollonius Euclidean ellipse to Lorentzian n-ellipse.

Defination 1. Let M be a surface in Lorentz 3-space. A Lorentzian n-ellipse g in M is a curve such that every point $P \in g$ has constant distance sum from n fixed points $F_1, \dots, F_n$, the so called foci or focal points of the n-ellipse, with three non collinear foci and $F_i \neq F_j, \forall i \neq j$.

Let λ_1, λ_2 and c_1 be three positive real numbers such that $\lambda_1 < \lambda_2$ and $c_1 < \min\left\{\lambda_1, \frac{\lambda_2}{2}\right\}$.

In the Lorentzian plane, the pure timelike curve g_1 given by

$$(x_1 + c_1)^2 - x_2^2 = \lambda_1^2$$

with $x_1 > 0$, is a Lorentzian 1-ellipse with focal point $F_1 = (-c_1, 0)$. Let us note that g_1 is a Lorentzian circle (see [2]).

In the same way, the pure timelike curve g_2 given by

$$\frac{x_1^2}{\lambda_2^2} - \frac{x_2^2}{\lambda_2^2 - 4c_1^2} = \frac{1}{4}$$

with $x_1 > 0$, is a Lorentzian 2-ellipse with focal points $F_1 = (-c_1, 0)$ and $F_2 = (c_1, 0)$. The equation of g_2 is equivalent to the equation of one of the ellipses obtained by Hano and Nomizu (cf. [5]).

In the following, $d(F_i, P)$ will be denoted by a_i, for all $i \geq 1$.

4. Lorentzian 3-ellipse

In this section, we will apply a geometric process to construct a Lorentzian 3-ellipse, g_3, such that the curve is a pure timelike curve and where the three focal points are the vertices of a pure spacelike triangle in the Lorentzian plane.

Let $F_1 = (-c_1, 0)$, $F_2 = (c_1, 0)$ and $F_3 = (c_2, b_2) \in L^2$, with $0 < c_2 < c_1$ and $0 < b_2 < \min\{c_2, c_1 - c_2\}$. Without loss of generality, we search a future-pointing timelike curve, g_3, such that $a_1 + a_2 + a_3 = \lambda_3$, for all $P \in g_3$, $\lambda_3 > 0$. Hence, the curve can be parameterized in the following way:

$$g_3(s) = \begin{cases} x_1(s) = f(s)\, ch(s), \\ x_2(s) = f(s)\, sh(s), \end{cases}$$

where s is the proper time parameter (i.e., the arc-length parameter for timelike curves), f is a function of a real variable such that $\left(\frac{df}{ds}(s)\right)^2 < f^2(s)$. Without loss of generality, we can consider $f > 0$.

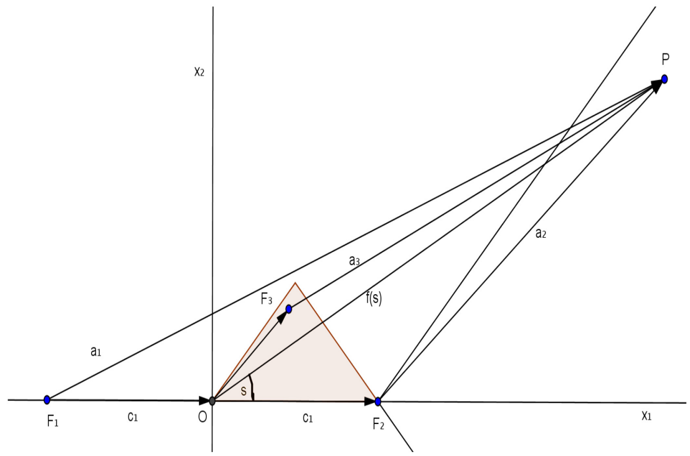

Figure 1. The triangles $T[F_1, O, P]$, $T[O, F_2, P]$ and $T[O, F_3, P]$.

Let us note that $T[F_1, O, P]$, $T[O, F_2, P]$, $T[O, F_3, P]$ and $T[F_1, F_3, F_2]$ are four pure spacelike triangles, where $O = (0, 0)$, (Figure 1).

Since $f(s) = d(O, P)$, $P \in g_3$, by the *Hyperbolic cosine law*, we have:

$$a_1^2 = c_1^2 + f^2(s) + 2c_1 f(s)\, ch(s), \tag{1}$$

$$a_2^2 = c_1^2 + f^2(s) - 2c_1 f(s)\, ch(s), \tag{2}$$

$$a_3^2 = \|\overrightarrow{OF_3}\|^2 + f^2(s) - 2\|\overrightarrow{OF_3}\| f(s)\, ch(\hat{\alpha}_3) \tag{3}$$

with $\hat{\alpha}_3$ the angle between $\overrightarrow{OF_3}$ and $\overrightarrow{OP}$, being F_3 the middle vertex of the future-pointing spacelike triangle $T[O, F_3, P]$.

Let λ_3 be a constant, $\lambda_3 > 0$, and $h_2 = h_2(P)$ such that $\begin{cases} \sum_{i=1}^{3} a_i = \lambda_3 \\ a_1 + a_2 = h_2 \\ 2c_1 < h_2 < \lambda_3 \end{cases}$ for all $P \in g_3$. Then, we have $\begin{cases} h_2 = a_1 + a_2 \\ \lambda_3 - h_2 = a_3 \end{cases}$, for all $P \in g_3$.

By (3), since $ch(\hat{\alpha}_3) = \dfrac{\langle \overrightarrow{OF_3}, \overrightarrow{OP} \rangle}{\|\overrightarrow{OF_3}\| . \|\overrightarrow{OP}\|}$, we have

$$\begin{aligned}(\lambda_3 - h_2)^2 = a_3^2 \\ &= \|\overrightarrow{OF_3}\|^2 + f^2(s) - 2\langle \overrightarrow{OF_3}, \overrightarrow{OP} \rangle \\ &= (c_2^2 - b_2^2) + (x_1^2(s) - x_2^2(s)) - 2(c_2 x_1(s) - b_2 x_2(s)) \\ &= (x_1^2(s) - 2c_2 x_1(s) + c_2^2) - (x_2^2(s) - 2b_2 x_2(s) + b_2^2) \\ &= (x_1(s) - c_2)^2 - (x_2(s) - b_2)^2 \qquad (4)\end{aligned}$$

for all $P \in g_3$. Therefore, we obtain the Lorentzian circle with center F_3 and radius $\lambda_3 - h_2$, with $f > 0$.

Let us note that for each h_2 fixed such that $2c_1 < h_2 < \lambda_3$, we have the Lorentzian 2-ellipse, i.e., the set of points $P \in L^2$ such that $a_1 + a_2 = h_2$.

By (1) and (2), we have

$$h_2^2 = a_1^2 + a_2^2 + 2a_1 a_2 = 2(c_1^2 + f^2(s)) + 2\sqrt{(c_1^2 + f^2(s))^2 - 4c_1^2 f^2(s) ch^2(s)},$$

where $f^2(s) = x_1^2(s) - x_2^2(s)$. By computing, we obtain the following equation

$$\frac{x_1^2(s)}{\frac{h_2^2}{4}} - \frac{x_2^2(s)}{\frac{h_2^2}{4} - c_1^2} = 1. \qquad (5)$$

Since we are looking for a Lorentzian 3-ellipse g_3 such that $\sum_{i=1}^{3} a_i = \lambda_3$, by (4) and (5), g_3 is given by the set of points $P = (f(s) ch(s), f(s) sh(s)) \in L^2$ such that $\left(\frac{df}{ds}(s)\right)^2 < f^2(s)$ and

$$\begin{cases} \dfrac{x_1^2(s)}{\dfrac{h_2^2(s)}{4}} - \dfrac{x_2^2(s)}{\dfrac{h_2^2(s)}{4} - c_1^2} = 1, \\ (x_1(s) - c_2)^2 - (x_2(s) - b_2)^2 = (\lambda_3 - h_2(s))^2, \end{cases} \tag{6}$$

with $\lambda_3 > 0$, $2c_1 < h_2(s) < \lambda_3$ and $-x_1(s) + c_1 < x_2(s) < x_1(s) - c_1$.

By symmetry $(x, y) \to (y, x)$, it is possible to construct a Lorentzian 3-ellipse which is a pure spacelike curve.

Therefore we start the following theorem.

Theorem 2. *Given* $\lambda_3 > 0$, *in the Lorentzian plane there exist a pure timelike curve* (*resp*., *pure spacelike curve*), g_3, *and three points* F_1, F_2, F_3 *such that* g_3 *is a Lorentzian* 3-*ellipse with* F_1, F_2, F_3 *vertices of a pure spacelike triangle* (*resp*., *pure timelike triangle*) *as foci*.

Remark 3. In the above construction, we consider the pure spacelike triangle $T[F_1, F_2, P]$ and we obtain that $2c_1 + a_2 \le a_1$, by the Reversed triangular inequality. Then $2c_1 < h_2$.

In addition, we note that $\begin{cases} \sqrt{\dfrac{h_2^2}{4} - c_1^2} + \dfrac{h_2}{2} < a_1 < h_2, \\ 0 < a_2 < \dfrac{h_2}{2} - \sqrt{c_1^2 - \dfrac{h_2^2}{4}}. \end{cases}$

4.1. A geometric interpretation of the 3-ellipse in L^3

According to [7], the $\mathbb{R}_1^3$ space with signature $(-, +, +)$ is congruent to L^3.

Let be $P = (x_1, x_2, x_3) \in \mathbb{R}_1^3$, $c_1 > 0$ and $\lambda_3 > 0$. The nullcone with vertice (c_2, b_2, λ_3), where $0 < c_2 < c_1$ and $0 < b_2 < \min\{c_2, c_1 - c_2\}$, has equation:

$$-(x_1 - c_2)^2 + (x_2 - b_2)^2 + (x_3 - \lambda_3)^2 = 0. \tag{7}$$

Let M be the surface in $\mathbb{R}_1^3$ given by

$$\begin{cases} \dfrac{x_1^2}{x_3^2} - \dfrac{x_2^2}{x_3^2 - 4c_1^2} = \dfrac{1}{4}, \\ 2c_1 < x_3 < \lambda_3. \end{cases} \tag{8}$$

By (8), we have

$[x_3^2 - 2(x_1^2 + c_1^2) + 2x_2^2]^2 - [-2(x_1^2 + c_1^2) + 2x_2^2]^2 - (-16c_1^2 x_1^2) = 0.$ Hence,

$$\left\|\left(\sqrt{2}\sqrt{x_1^2 + c_1^2},\ \sqrt{2}\, x_2,\ x_3\right)\right\|^4 - \left\|\left(\sqrt{2}\sqrt{x_1^2 + c_1^2},\ \sqrt{2}\, x_2,\ 0\right)\right\|^4 = -16c_1^2 x_1^2.$$

Then the curve $\tilde{g}_3 \subset \mathbb{R}_1^3$ satisfying

$$\begin{cases} \left\|\left(\sqrt{2}\sqrt{x_1^2 + c_1^2},\ \sqrt{2}\, x_2,\ x_3\right)\right\|^4 - \left\|\left(\sqrt{2}\sqrt{x_1^2 + c_1^2},\ \sqrt{2}\, x_2,\ 0\right)\right\|^4 = -16c_1^2 x_1^2, \\ \left\|(x_1 - c_2,\ x_2 - b_2,\ x_3 - \lambda_3)\right\|^2 = 0, \\ 2c_1 < x_3 < \lambda_3 \end{cases}$$

is identified with the curve g_3 when $x_3 = h_2$.

5. Lorentzian 4-ellipse

Here, we show a geometric process to construct a Lorentzian 4-ellipse, g_4, such that the curve is a pure timelike curve and where the four focal points are the vertices of a pure spacelike quadrilateral in the Lorentzian plane.

Let $F_1 = (-c_1, 0)$, $F_2 = (c_1, 0)$, $F_3 = (c_2, b_2)$ and $F_4 = (-c_2, b_2) \in L^2$, with $0 < c_2 < c_1$ and $0 < b_2 < \min\{c_2, c_1 - c_2\}$. Without loss of generality, we search a future-pointing timelike curve, g_4, such that $a_1 + a_2 + a_3 + a_4 = \lambda_4$, for all $P \in g_4$, $\lambda_4 > 0$. Hence, the curve can be parameterized in the following way:

$$g_4(s) = \begin{cases} x_1(s) = f(s)\, ch(s), \\ x_2(s) = f(s)\, sh(s), \end{cases}$$

where s is the proper time parameter, f is a function of a real variable such that

$\left(\frac{df}{ds}(s)\right)^2 < f^2(s)$. Without loss of generality, we can consider $f > 0$.

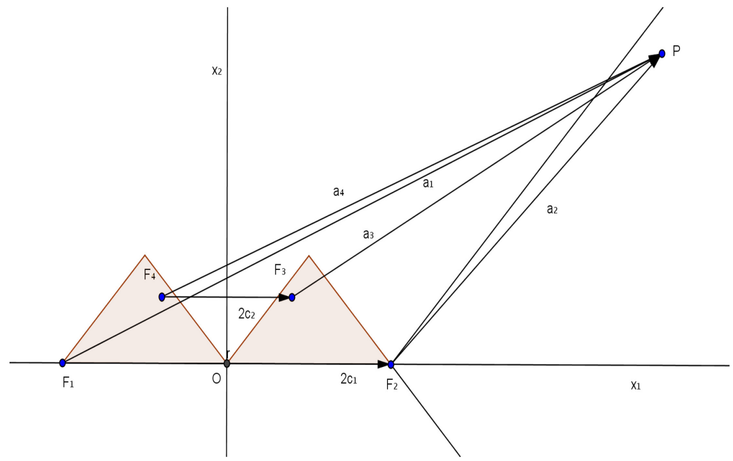

Figure 2. The triangles $T[F_1, F_2, P]$ and $T[F_4, F_3, P]$.

Let us note that $T[F_1, F_2, P]$ and $T[F_4, F_3, P]$ are pure spacelike triangles, (Figure 2). Also $\mathbf{P}[F_1, F_4, F_3, F_2]$ is a pure spacelike quadrilateral.

Let λ_4 be a constant, $\lambda_4 > 0$, and $h_4 = h_4(P)$ such that $\begin{cases} \sum_{i=1}^{4} a_i = \lambda_4 \\ a_3 + a_4 = h_4 \\ 2c_2 < h_4 < \lambda_4 \end{cases}$, for all $P \in g_4$. Then, we have $\begin{cases} h_4 = a_3 + a_4 \\ \lambda_4 - h_4 = a_1 + a_2 \end{cases}$, for all $P \in g_4$.

Let us remark that the condition $h_4 > 2c_2$ is derived from the Reversed triangular inequality applied to $T[F_4, F_3, P]$.

Since $\lambda_4 - h_4 = a_1 + a_2$, by (1) and (2), we have

$$(\lambda_4 - h_4)^2 = (a_1 + a_2)^2$$

$$= 2(c_1^2 + f^2(s)) + 2\sqrt{(c_1^2 + f^2(s))^2 - 4c_1^2 f^2(s)\, ch^2(s)},$$

for all $P \in g_4$.

Analogously to Section 4, for each h_4 fixed such that $\begin{cases} h_4 > 2c_2 \\ \lambda_4 - h_4 > 2c_1 \end{cases}$, we obtain

$$\frac{x_1^2(s)}{\frac{(\lambda_4 - h_4)^2}{4}} - \frac{x_2^2(s)}{\frac{(\lambda_4 - h_4)^2}{4} - c_1^2} = 1, \tag{9}$$

for all $P \in g_4$, which is a Lorentzian 2-ellipse with focus F_1 and F_2, with $\lambda_4 - h_4 > 2c_1$.

Now, we consider the pure spacelike future-pointing triangle $T[F_4, F_3, P]$. Let $d_2 = d(O', P)$, $P \in g_4$, with $O' = (0, b_2)$. By the *Hyperbolic cosine law*, we have:

$$a_3^2 = c_2^2 + d_2^2 - 2c_2 d_2\, ch(\hat{t}), \tag{10}$$

$$a_4^2 = c_2^2 + d_2^2 + 2c_2 d_2\, ch(\hat{t}), \tag{11}$$

where $\hat{t}$ is the angle between $\overrightarrow{O'F_3}$ and $\overrightarrow{O'P}$.

By (10) and (11), we have

$$\begin{aligned} h_4^2 &= (a_3 + a_4)^2 \\ &= 2(c_2^2 + d_2^2) + 2\sqrt{(c_2^2 + d_2^2)^2 - 4c_2^2 d_2^2 ch^2(\hat{t})}, \end{aligned}$$

for all $P \in g_4$, $d_2^2 = x_1^2(s) - (x_2(s) - b_2)^2$ and $ch^2(\hat{t}) = \dfrac{x_1^2(s)}{x_1^2(s) - (x_2(s) - b_2)^2}$.

Therefore, for each h_4 fixed, we have

$$\frac{x_1^2(s)}{\frac{h_4^2}{4}} - \frac{(x_2(s) - b_2)^2}{\frac{h_4^2}{4} - c_2^2} = 1, \tag{12}$$

for all $P \in g_4$, which is a Lorentzian 2-ellipse with foci F_3 and F_4, where $h_4 > 2c_2$.

By (9) and (12), the future-pointing timelike curve g_4, is given by the set of points $P = (f(s)ch(s),\ f(s)sh(s)) \in L^2$ such that $\left(\frac{df}{ds}(s)\right)^2 < f^2(s)$ and

$$\begin{cases} \dfrac{x_1^2(s)}{\dfrac{h_4^2(s)}{4}} - \dfrac{(x_2(s) - b_2)^2}{\dfrac{h_4^2(s)}{4} - c_2^2} = 1, \\ \dfrac{x_1^2(s)}{\dfrac{(\lambda_4 - h_4(s))^2}{4}} - \dfrac{x_2^2(s)}{\dfrac{(\lambda_4 - h_4(s))^2}{4} - c_1^2} = 1 \end{cases} \tag{13}$$

with $h_4 > 2c_2$, $\lambda_4 - h_4 > 2c_1$ and $-x_1(s) + c_2 < x_2(s) < x_1(s) - c_2$.

By symmetry $(x,\ y) \to (y,\ x)$, it is possible to construct a Lorentzian 4-ellipse which is a pure spacelike curve.

Therefore we start the following theorem.

Theorem 4. *Given* $\lambda_4 > 0$, *in the Lorentzian plane there exist a pure timelike curve* (*resp*., *pure spacelike curve*), g_4, *and four points* F_1, F_2, F_3, F_4 *such that* g_4 *is a Lorentzian* 4-*ellipse with* F_1, F_2, F_3, F_4 *vertices of a pure spacelike quadrilateral* (*resp*., *pure timelike quadrilateral*) *as foci*.

5.1. A geometric interpretation of the 4-ellipse in L^3

Let M be the surface in $\mathbb{R}_1^3$ given by:

$$\begin{cases} \dfrac{x_1^2}{x_3^2} - \dfrac{(x_2 - b_2)^2}{x_3^2 - 4c_2^2} = \dfrac{1}{4}, \\ 2c_1 < x_3 < \lambda_4;\ 0 < c_2 < c_1;\ 0 < b_2 < \min\{c_2,\ c_1 - c_2\}. \end{cases} \tag{14}$$

By (14), we have:

$$[x_3^2 - 2(c_2^2 + x_1^2) + 2(x_2 - b_2)^2]^2 - [2(c_2^2 + x_1^2) - 2(x_2 - b_2)^2]^2$$
$$-(-16c_2^2 x_1^2) = 0.$$

Hence

$$\left\|\left(\sqrt{2}\sqrt{c_2^2 + x_1^2},\ \sqrt{2}(x_2 - b_2),\ x_3\right)\right\|^4$$

$$-\left\|\left(\sqrt{2}(x_2 - b_2),\ \sqrt{2}\sqrt{c_2^2 + x_1^2},\ 0\right)\right\|^4 = -16c_2^2x_1^2. \tag{15}$$

On the other hand, let N be the surface in $\mathbb{R}_1^3$ given by

$$\begin{cases} \dfrac{x_1^2}{(\lambda_4 - x_3)^2} - \dfrac{x_2^2}{(\lambda_4 - x_3)^2 - 4c_1^2} = \dfrac{1}{4}, \\ 2c_1 < \lambda_4 - x_3. \end{cases} \tag{16}$$

By (16), we have

$$\left\|\left(\sqrt{2}\sqrt{c_1^2 + x_1^2},\ \sqrt{2}x_2,\ \lambda_4 - x_3\right)\right\|^4 - \left\|\left(\sqrt{2}\sqrt{c_1^2 + x_1^2},\ \sqrt{2}x_2,\ 0\right)\right\|^4$$

$$= -16c_1^2x_1^2. \tag{17}$$

Then, by (15) and (17), the curve $\tilde{g}_4$ satisfying

$$\begin{cases} \left\|\left(\sqrt{2}\sqrt{c_2^2 + x_1^2}, \sqrt{2}(x_2 - b_2), x_3\right)\right\|^4 - \left\|\left(\sqrt{2}(x_2 - b_2), \sqrt{2}\sqrt{c_2^2 + x_1^2}, 0\right)\right\|^4 = -16c_2^2x_1^2, \\ \left\|\left(\sqrt{2}\sqrt{c_1^2 + x_1^2}, \sqrt{2}x_2, \lambda_4 - x_3\right)\right\|^4 - \left\|\left(\sqrt{2}\sqrt{c_1^2 + x_1^2}, \sqrt{2}x_2, 0\right)\right\|^4 = -16c_1^2x_1^2, \\ 2c_1 < \min\{x_3, \lambda_4 - x_3\}; 0 < c_2 < c_1; 0 < b_2 < \min\{c_2, c_1 - c_2\} \end{cases}$$

is identified with the curve g_4, when $x_3 = h_4$.

6. Main Result

A Lorentzian 5-ellipse and a Lorentzian 6-ellipse can be constructed in the same way to Lorentzian 3-ellipse and to Lorentzian 4-ellipse, respectively. Thus, we aplly these two geometric process to construct the Lorentzian n-ellipses both for n even and for n odd.

Theorem 5. *Given* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m$, *in the Lorentzian plane there exist* *families* $\{g_n\}_{n\in\{1,\ldots,m\}}$ *of pure timelike curves* (*resp*., *pure spacelike curves*) *and* $F_1, \ldots, F_m$ *points such that for all* $n \in \{1, \ldots, m\}$, g_n *is a Lorentzian n-ellipse with* $F_1, \ldots, F_n$ *vertices of a pure spacelike polygon* (*resp*., *pure timelike polygon*) *as foci*, *and*

$$\sum_{i=1}^{n} d(F_i, P) = \lambda_n,$$

for all $P \in g_n$.

Proof. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_m$, where λ_i is a positive real number for all i.

In the following, we show a family $\{g_n\}_{n\in\{1,\ldots,m\}}$ of n-ellipses satisfying the above mentioned conditions. In order to do it, without loss of generality, we fix m points $F_1, \ldots, F_m$ in L^2 as it is shown in Table 1.

Table 1. Choice of focal points

Points	Conditions
$F_1 = (-c_1, 0)$ $F_2 = (c_1, 0)$	$0 < c_1 < \min\left\{\lambda_1, \frac{\lambda_2}{2}\right\}$
$F_3 = (c_2, b_2)$ $F_4 = (-c_2, b_2)$	$0 < c_2 < c_1$ $0 < b_2 < \min\{c_2, c_1 - c_2\}$
……………………… and for all $k \geq 2$ $F_{2k+1} = (c_{k+1}, b_{k+1})$ $F_{2(k+1)} = (-c_{k+1}, b_{k+1})$	…………………………….. $b_k < c_{k+1} < c_k$ $0 < b_{k+1} < \min\{c_{k+1}, c_k - c_{k+1}\}$

Hence, $\mathbf{P}[F_1, F_2, \ldots, F_{2(k+1)}]$ and $\mathbf{P}[F_1, F_2, \ldots, F_{2k+1}]$ are future-pointing spacelike polygons for all k in the set of nonnegative integer numbers. Let us recall that $\mathbf{P}[F_1]$ and $\mathbf{P}[F_1, F_2]$ are called degenerate polygons.

By using the geometric process applied in above sections, we construct the family $\{g_n\}_{n\in\{1,\ldots,m\}}$ of Lorentzian n-ellipses, where $g_n = g_n(s)$ and for each n, s denote the proper time parameter.

In order to do it, we define auxiliary functions $h_{2k}(s) = a_{2k-1} + a_{2k}$ for all $k \geq 1$.

Thus, we obtain the following family of pure timelike curves g_n, for all $n \in \{1, ..., m\}$, summarised in Table 2,

Table 2. Equations of Lorentzian n-ellipses

n foci	$g_n(s) = (x_1(s), x_2(s))$
F_1	$(x_1(s) + c_1)^2 - x_2^2(s) = \lambda_1^2$
F_1, F_2	$\frac{x_1^2(s)}{\lambda_2^2(s)} - \frac{x_2^2(s)}{\lambda_2^2(s) - 4c_1^2} = \frac{1}{4}$
F_1, F_2, F_3	$\begin{cases} \frac{x_1^2(s)}{h_2^2(s)} - \frac{x_2^2(s)}{h_2^2(s) - 4c_1^2} = \frac{1}{4}, \\ (x_1(s) - c_2)^2 - (x_2(s) - b_2)^2 = (\lambda_3 - h_2(s))^2 \end{cases}$
F_1, F_2, F_3, F_4	$\begin{cases} \frac{x_1^2(s)}{h_4^2(s)} - \frac{(x_2(s) - b_2)^2}{h_4^2(s) - 4c_2^2} = \frac{1}{4}, \\ \frac{x_1^2(s)}{(\lambda_4 - h_4(s))^2} - \frac{x_2^2(s)}{(\lambda_4 - h_4(s))^2 - 4c_1^2} = \frac{1}{4} \end{cases}$
….................	…………………………….
$F_1, ..., F_{2k+1}$	$\begin{cases} \frac{x_1^2(s)}{h_2^2(s)} - \frac{x_2^2(s)}{h_2^2(s) - 4c_1^2} = \frac{1}{4}, \\ \vdots \\ \frac{x_1^2(s)}{h_{2k}^2(s)} - \frac{(x_2(s) - b_k)^2}{h_{2k}^2(s) - 4c_k^2} = \frac{1}{4}, \\ (x_1(s) - c_{k+1})^2 - (x_2(s) - b_{k+1})^2 = \left(\lambda_{2k+1} - \sum_{j=1}^{k} h_{2j}(s)\right)^2 \end{cases}$
$F_1, ..., F_{2(k+1)}$	$\begin{cases} \frac{x_1^2(s)}{h_4^2(s)} - \frac{(x_2(s) - b_2)^2}{h_4^2(s) - 4c_2^2} = \frac{1}{4}, \\ \vdots \\ \frac{x_1^2(s)}{h_{2(k+1)}^2(s)} - \frac{(x_2(s) - b_{k+1})^2}{(h_{2(k+1)}^2(s) - 4c_{k+1}^2} = \frac{1}{4}, \\ \frac{x_1^2(s)}{\left(\lambda_{2(k+1)} - \sum_{j=2}^{k+1} h_{2j}(s)\right)^2} - \frac{x_2^2(s)}{\left(\lambda_{2(k+1)} - \sum_{j=2}^{k+1} h_{2j}(s)\right)^2 - 4c_1^2} = \frac{1}{4} \end{cases}$

where

$$\begin{cases} \lambda_{2k+1} = \sum_{i=1}^{2k+1} d(P(s), F_i), \\ h_{2j}(s) = d(P(s), F_{2j-1}) + d(P(s), F_{2j}), \forall j \in \{1, ..., k\} \quad \text{if } n = 2k+1, \\ h_{2j}(s) > 2c_j, \forall j \in \{1, ..., k\}, \end{cases}$$

and

$$\begin{cases} \lambda_{2(k+1)} = \sum_{i=1}^{2k+2} d(P(s), F_i), \\ h_{2j}(s) = d(P(s), F_{2j-1}) + d(P(s), F_{2j}), \forall j \in \{2, ..., k+1\} \\ \qquad \text{if } n = 2(k+1), \\ h_{2j}(s) > 2c_j, \forall j \in \{2, ..., k+1\}, \\ \left(\lambda_{2(k+1)} - \sum_{j=2}^{k+1} h_{2j}(s)\right) > 2c_1. \end{cases}$$

In addition, by symmetry $(x, y) \to (y, x)$, we found a family $\{g_n\}_{n \in \{1, ..., m\}}$ of Lorentzian n-ellipses where for all $n \in \{1, ..., m\}$, g_n is a pure spacelike curve with $F_1, ..., F_n$ vertices of a pure timelike polygon as foci, and

$$\sum_{i=1}^{n} d(F_i, P) = \lambda_n$$

for all $P \in g_n$.

References

[1] G. Birman and G. Desideri, Projections of pseudoesphere in the Lorentz 3-space, Bull. Korean Math. Soc. 44(3) (2007), 483-492.

[2] G. Birman and K. Nomizu, Trigonometry in Lorentzian geometry, Amer. Math. Monthly 91(9) (1984), 43-549.

[3] R. Bix, Conics and Cubics: A Concrete Introduction to Algebraic Curves, Springer Science and Business Media, New York, 1998.

[4] G. Desideri, On polygons in Lorentzian plane, Comptes rendus de l'Academie bulgare des Sciences 60(10) (2007), 1039-1045.

[5] J. Hano and K. Nomizu, Surfaces of revolution with constant mean curvature in Lorentz-Minkowski space, Tohoku Math. J. 36 (1984), 427-437.

[6] M. J. Khilji, Multi foci closed curves, J. Theoretics 6(6) (2004).

[7] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

[8] M. Petrović, B. Banjac and B. Malešević, The geomerty of trifocal curves with applications in architecture, urban and spatial planning, Spatium International Review 32 (2014), 28-33.