Fundamental Journal of Mathematics and Mathematical Sciences
Vol. 7, Issue 1, 2017, Pages 59-72
This paper is available online at http://www.frdint.com/
Published online March 12, 2017

INVESTIGATION OF THE NEW CLASS OF THE NONLINEAR RATIONAL DIFFERENCE EQUATIONS

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Abstract

In this paper, we consider the asymptotic behavior of the solution of the new class of the nonlinear rational difference equations. We consider the local, global stability and boundedness of the solution. Moreover, we investigate the new periodic character of periodic two of solutions of these equations, which is not familiar. We do not know a similar approach of periodic for other class of the nonlinear rational difference equations. Moreover, we give some interesting counter examples in order to verify our strong results.

Keywords and phrases: difference equations, equilibrium points, local and global stability, boundedness, periodic solution.

2010 Mathematics Subject Classification: 39A10, 39A23, 39A30, 39A99.

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Received December 19, 2016; Accepted February 18, 2017

1. Introduction

The difference equations describe real life situations in probability theory, statistical problems, queuing theory, electrical network, combinatorial analysis, sociology, psychology, genetics in biology, economics, etc., see, e.g., [13, 14]. There are many works about the global asymptotic of solutions of rational difference equations, [15-19] and references therein. It is very important to investigate the asymptotic behavior of solutions of a system of nonlinear difference equations and to discuss the boundedness, periodicity and stability (local and global) of their equilibrium points.

El-Owaidy et al. [8] studied the asymptotic behavior of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}^{p}}{x_{n}^{p}}.$$

Elabbasy et al. [6] studied the behavior of solutions of a class of nonlinear rational difference equation

$$x_{n+1} = \alpha x_{n-k} + \frac{\beta x_{n-\ell}^{\delta}}{\gamma x_{n-s}^{\delta}}.$$

For more investigation of the asymptotic behaviour of solutions of rational difference equations, one can refer to [1-12] and references therein.

In this paper, we consider analytical investigation of the solution of the following recursive sequence

$$x_{n+1} = a x_{n-k} + \delta\left(\frac{b x_{n-l}}{c x_{n-k} + d x_{n-l}}\right)^{\alpha}, \tag{1.1}$$

where the initial conditions $x_{-r}, x_{-r+1}, \ldots, x_{-1}, x_{0}$, $r = \max\{l, k\}$ are arbitrary real numbers and a, b, c, d, α are constant real numbers.

In this section, we present the basic definitions and theorems of our model, namely equilibrium points, local and global stability, boundedness, periodicity and the oscillation of the solution.

Definition 1.1 (Equilibrium point)**.** Consider a difference equation in the form

$$x_{n+1} = F(x_{n-l}, x_{n-k}), \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where F is a continuous function, while l and k are positive integers. A point $\bar{x}$ is said to be an equilibrium point of the equation (1.1) if it is a fixed point of F, i.e., $\bar{x} = F(\bar{x}, \bar{x})$.

Definition 1.2 [16] (stability)**.** Let $\bar{x} \in (0, \infty)$ be an equilibrium point of equation (1.2). Then we have

(a) Local stability

An equilibrium point $\bar{x}$ of equation (1.2) is said to be locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-\nu} \in (0, \infty)$ for $\nu = 0, 1, \ldots, r$ with

$$\sum_{i=0}^{r} |x_{-i} - \bar{x}| < \delta,$$

then $|x_{-n} - \bar{x}| < \varepsilon$ for all $n \geq -r$.

(b) Global stability

An equilibrium point $\bar{x}$ of equation (1.2) is said to be a global attractor if for every $x_{-\nu} \in (0, \infty)$ for $\nu = 0, 1, \ldots, r,$ we have

$$\lim_{n \to \infty} x_n = \bar{x}.$$

(c) Global asymptotic stability

An equilibrium point $\bar{x}$ of equation (1.2) is said to be globally asymptotically stable if it is locally stable and a global attractor.

(d) Unstability

An equilibrium point $\bar{x}$ of equation (1.2) is said to be unstable if it is not locally stable.

Definition 1.3 [16] (Periodicity)**.** A sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be periodic with period t if $x_{n+t} = x_n$ for all $n \geq -r$. A sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be periodic with prime period t if t is the smallest positive integer having this property.

Definition 1.4 [16] (Boundedness)**.** Equation (1.2) is called permanent and bounded if there exist numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $x_{-\nu} \in (0, \infty)$ for $\nu = 0, 1, \ldots, r$ there exists a positive integer N which depends on these initial conditions such that $0 < m < M < \infty$ for all $n \geq N$.

Definition 1.5 [16]**.** The linearized equation of equation (1.2) about the equilibrium point $\bar{x}$ is defined by the linear difference equation

$$y_{n+1} = c_0 y_{n-1} + c_1 y_n, \tag{1.3}$$

where

$$c_i = \frac{\partial F(\bar{x}, \bar{x})}{\partial x_{n-i}}, \; i = 0, 1.$$

Theorem 1.1 [14]**.** *Assume that* $h_0, h_1 \in R$. *Then*

$$|c_0| + |c_1| < 1,$$

is a sufficient condition for the asymptotic stability of equation (1.2).

The rest of the paper is organized as follows: In Section 2, we study the asymptotic behavior of the solution of the new class of the nonlinear rational difference equations. We present the local and global stability of the solution for equation (1.1) and give an interesting counter example to support our analysis. We also prove that the positive solution of equation (1.1) is bounded. In Section 3, we study the periodic behaviour of the solution for the equation (1.1). We also give two counter examples to show how our model is so rich.

2. Dynamics of Equation (1.1)

This section is concerned with the asymptotic behavior of the solution of the new class of the nonlinear rational difference equations, namely the local, global stability and boundedness of the solution.

2.1. The stability of solutions

Here we study the local stability of the equilibrium point of equation (1.1). The positive equilibrium point of equation (1.1) is given by

$$\bar{x} = \frac{\delta}{1-a}\left(\frac{b}{c+d}\right)^{\alpha}.$$

Now, we define a continuous function $h : (0, \infty)^2 \to (0, \infty)$ such that

$$f(u, v) = au + \delta\left(\frac{bv}{cu+dv}\right)^{\alpha}. \tag{2.1}$$

Therefore, it follows that

$$\begin{aligned}\frac{\partial f}{\partial u}(u, v) &= a + \delta\alpha\left(\frac{bv}{cu+dv}\right)^{\alpha-1}\frac{-cbv}{(cu+dv)^2}\\ &= a - \delta\alpha\frac{c}{cu+dv}\left(\frac{bv}{cu+dv}\right)^{\alpha}\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\frac{\partial f}{\partial v}(u, v) &= \delta\alpha\left(\frac{bv}{cu+dv}\right)^{\alpha-1}\frac{b(cu+dv)-dbv}{(cu+dv)^2}\\ &= \delta\alpha\frac{cu}{v(cu+dv)}\left(\frac{bv}{cu+dv}\right)^{\alpha}.\end{aligned} \tag{2.3}$$

2.1.1. Local stability of equilibrium point

Theorem 2.1. *The positive equilibrium point of equation* (1.1) *is locally asymptotically stable if*

$$|a(c+d) - c\alpha(1-a)| + |c\alpha(1-a)| < c + d. \tag{2.4}$$

Proof. From equations (2.2) and (2.3), we see that

$$\left.\frac{\partial f}{\partial u}\right|_{(\bar{x}, \bar{x})} = a - c\alpha\frac{1-a}{c+d} = p_u,$$

and

$$\left.\frac{\partial f}{\partial v}\right|_{(\bar{x}, \bar{x})} = c\alpha\frac{1-a}{c+d} = p_v.$$

Thus, the linearized equation of (1.1) about the equilibrium point $\bar{x}$ is the linear difference equation

$$y_{n+1} = p_u y_{n-k} + p_v y_{n-l}.$$

It is follows by Theorem 1.1 that, equation (1.1) is locally stable if

$$|p_u| + |p_v| = \left|a - c\alpha \frac{1-a}{c+d}\right| + \left|c\alpha \frac{1-a}{c+d}\right| < 1$$

and so

$$|a(c+d) - c\alpha(1-a)| + |c\alpha(1-a)| < c+d.$$

Hence, the proof is complete.

Remark 2.2. If $a < 1$ and $a(c+d) > c\alpha(1-a)$, then condition (2.4) holds. Also, if $a < 1$ and $a(c+d) < c\alpha(1-a)$, then condition (2.4) becomes $2c\alpha(1-a) < (c+d)(1+a)$. On the other hand, if $a > 1$, then condition (2.4) is not satisfied and hence the equilibrium point of equation (1.1) is unstable.

Example 2.1. Consider the equation

$$x_{n+1} = \frac{1}{2} x_{n-4} + \left(\frac{x_{n-1}}{x_{n-4} + x_{n-1}}\right)^{1/2}. \tag{2.5}$$

We note that $\alpha = 0.5$, $k = 4$, $l = 1$ and $\delta = 1$. By Theorem 2.1, the positive equilibrium point $\bar{x} = 1.4142$ is locally asymptotically stable. Also, let the equation

$$x_{n+1} = \frac{1}{5} x_n - \left(\frac{x_{n-1}}{x_n + 5x_{n-1}}\right)^3. \tag{2.6}$$

We note that $\alpha = 3$, $k = 0$, $l = 1$ and $\delta = -1$. By Theorem 2.1, the positive equilibrium point $\bar{x} = -0.0058$ is locally asymptotically stable. On the other hand, consider

$$x_{n+1} = 1.1x_n + \frac{x_{n-1}}{x_n + x_{n-1}}. \tag{2.7}$$

We note that the positive equilibrium point of those equations is unstable.

2.1.2. Global attractivity of equilibrium point

In this section, we investigate the global asymptotic stability of equation (1.1) when $\alpha = 1$ or 2.

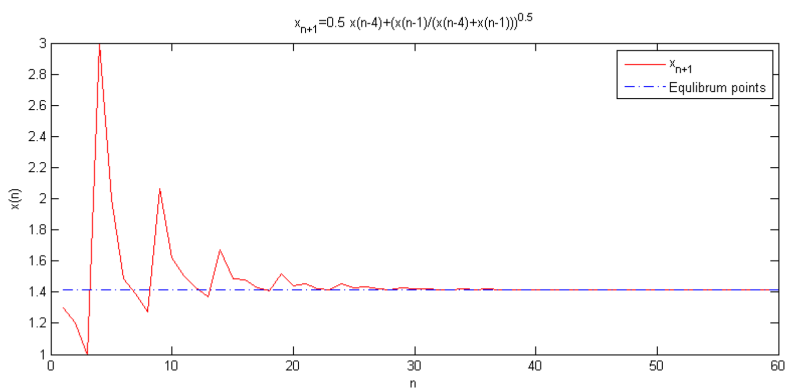
Theorem 2.3. *If* $\delta = -1$ *and* $a > 1$, *then the equilibrium point* $\bar{x}$ *is a global attractor of equation* (1.1).

Proof. By equations (2.2) and (2.3), we see that a function *f* defined as (1.2) is increasing in *u* and decreasing in *v*. Next, suppose that (m, M) is a solution of the system

$$m = f(m, M) \quad \text{and} \quad M = f(M, m).$$

Assume that $\alpha = 1$. Thus, by equation (1.1), we find

$$(1-a)m = -\frac{bM}{cm + dM} \quad \text{and} \quad (1-a)M = -\frac{bm}{cM + dm}.$$



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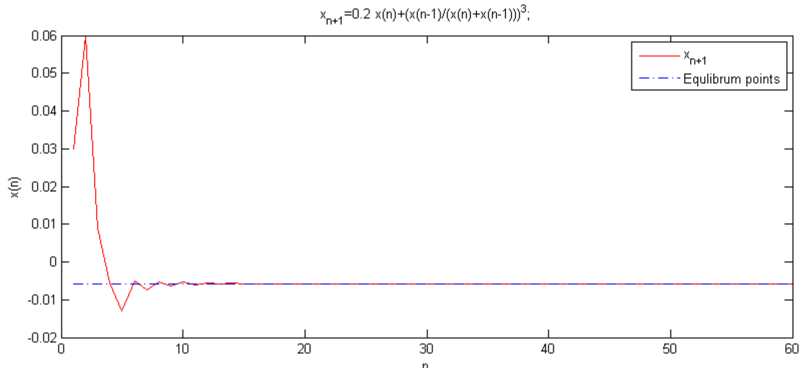

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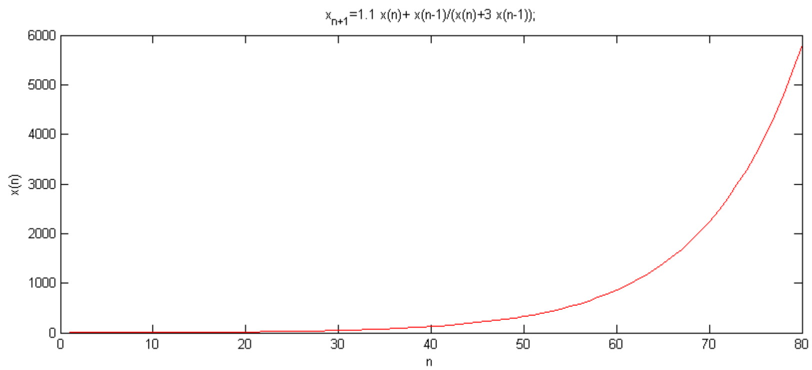

(III)

Figure 1. The stable and unstable solutions corresponding to differences equations (2.5), (2.6) and (2.7), respectively.

Hence

$$(M - m)\left((1 - a) - \frac{bc(M + m)}{M^2cd + Mc^2m + Md^2m + cdm^2}\right) = 0.$$

Then, we get $M = m$. It follows by [14, Theorem 1.4.5] that $\bar{x}$ is a global attractor of equation (1.1). In the case where $\alpha = 2$, from equation (1.1), we find

$$(1 - a)m = -\left(\frac{bM}{cm + dM}\right)^2 \quad \text{and} \quad (1 - a)M = -\left(\frac{bm}{cM + dm}\right)^2.$$

Hence

$$(M - m)\left((1-a) - b^2 c(M + m)\frac{cM^2 + 2dMm + cm^2}{(M^2 cd + Mc^2 m + Md^2 m + cdm^2)^2}\right) = 0.$$

Then, we have $M = m$. It follows by [14, Theorem 1.4.5] that $\bar{x}$ is a global attractor of equation (1.1) and then the proof is completed.

2.2. Boundedness of the solutions

In this section, we investigate the boundedness of the positive solutions of equation (1.1).

Theorem 2.4. *Assume that* $\delta = 1$ *and* $\{x_n\}_{n=-t}^{\infty}$ *be a solution of equation* (1.1) *where* $t = \max\{k, l\}$. *Then* $\{x_n\}_{n=-t}^{\infty}$ *is bounded if* $a < 1$ *and unbounded if* $a > 1$.

Proof. Let $\{x_n\}_{n=-\max\{k,l\}}^{\infty}$ be a solution of equation (1.1). It follows from equation (1.1) that

$$\begin{aligned} x_{n+1} &= ax_{n-k} + \left(\frac{bx_{n-l}}{cx_{n-k} + dx_{n-l}}\right)^{\alpha} \\ &\leq ax_{n-k} + \left(\frac{b}{d}\right)^{\alpha}. \end{aligned}$$

By using a comparison, we can write the right hand side as follows

$$y_{n+1} = ay_{n-k} + \left(\frac{b}{d}\right)^{\alpha}.$$

Then, we obtain

$$y_n = a^{\frac{1}{k+1}n} y_{-k} + \frac{1}{(1-a)}\left(\frac{b}{d}\right)^{\alpha}$$

and this equation is locally stable because $a < 1$ and converges to the equilibrium point

$$\bar{y} = \frac{1}{(1-a)}\left(\frac{b}{d}\right)^{\alpha}.$$

Therefore, we have

$$\limsup_{n\to\infty} x_n \leq \frac{1}{(1-a)}\left(\frac{b}{d}\right)^{\alpha}.$$

Hence $\{x_n\}_{n=-t}^{\infty}$ is bounded. On the other hand, from (1.1), we see that $x_{n+1} > ax_{n-k}$. If we set $z_{n+1} = az_{n-k}$, then $z_n = a^{\frac{1}{k+1}n} z_{-k}$ and z_n is unstable because $a > 1$. Therefore, $\limsup_{n\to\infty} x_n = \infty$ and hence the proof is complete.

3. Periodic Solution of Period Two

The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

Theorem 3.1. *Assume that k and l odd, $a \neq -1$. Then equation* (1.1) *has no a prime period two solution.*

Proof. Suppose that there exists a prime period two solution of equation (1.1)

$$\ldots,\ p,\ q,\ p,\ q,\ \ldots.$$

Thus, from (1.1), we get that

$$(1-a)p = \delta\left(\frac{b}{c+d}\right)^{\alpha} \quad \text{and} \quad (1-a)q = \delta\left(\frac{b}{c+d}\right)^{\alpha}.$$

Let $p = nq$ and $n \neq 0, \pm 1$, then, we get

$$(1-a)(p-nq) = (1-n)\delta\left(\frac{b}{c+d}\right)^{\alpha}$$

$$= 0$$

which is a contradiction, and hence the proof is complete.

Example 3.1. Consider the difference equation

$$x_{n+1} = -2x_{n-4} + \left(\frac{x_{n-2}}{x_{n-4} + x_{n-2}}\right)^{3} \tag{3.1}$$

with initial condition $x_1 = 2.0979,\ x_2 = -1.9729,\ x_3 = 2.0979,\ x_4 = -1.9729.$

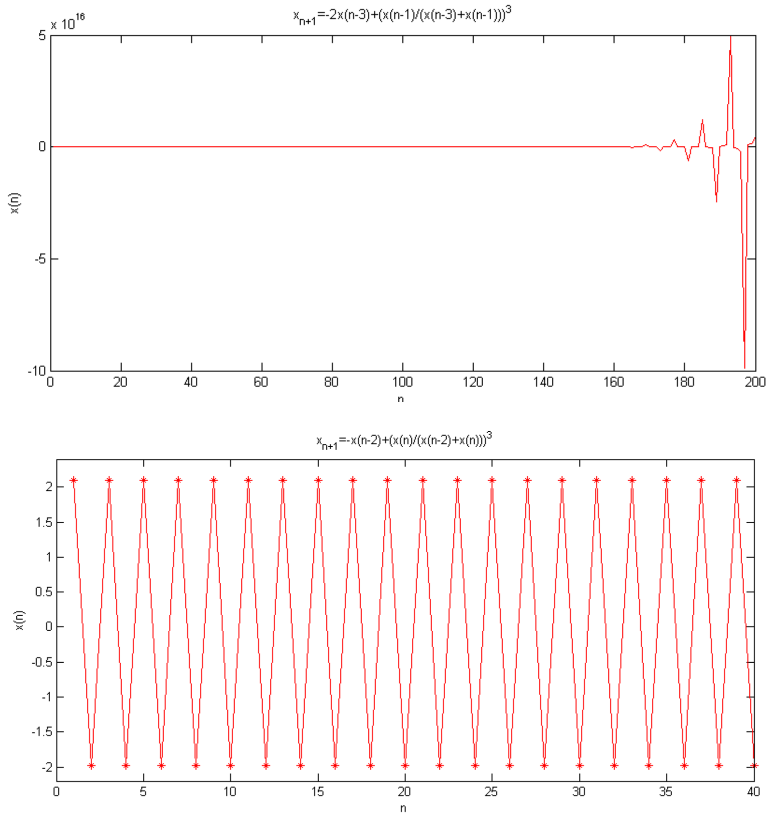

Figure 2. No prime and prime solutions corresponding to differences equations (3.1) and (3.2), respectively.

Theorem 3.2. *Assume that k and l are even. If $a = -1$, then equation* (1.1) *has a prime period two solution.*

Proof. Suppose that there exists a prime period two solution of equation (1.1)

$$\ldots,\ p,\ q,\ p,\ q,\ \ldots .$$

Thus, from (1.1), we get that

$$p = aq + \delta\left(\frac{b}{c+d}\right)^{\alpha} \quad \text{and} \quad q = ap + \delta\left(\frac{b}{c+d}\right)^{\alpha}.$$

Thus, we obtain

$$p = a^2 p + (a+1)\delta\left(\frac{b}{c+d}\right)^{\alpha},$$

$$q = a^2 q + (a+1)\delta\left(\frac{b}{c+d}\right)^{\alpha}.$$

Let $p = nq$ and $n \neq 0, \pm 1,$ then, we get

$$p - nq = a^2(p - nq) + (1-n)(a+1)\delta\left(\frac{b}{c+d}\right)^{\alpha}.$$

By using the fact $p - nq = 0,$ we obtain

$$(1-n)(a+1)\delta\left(\frac{b}{c+d}\right)^{\alpha} = 0.$$

Since $n \neq 1$ and $b \neq 0,$ we have that $a = -1.$ The proof is complete.

Example 3.2. Consider the difference equation

$$x_{n+1} = -x_{n-3} + \left(\frac{x_{n-1}}{x_{n-3} + x_{n-1}}\right)^3 \tag{3.2}$$

with initial condition $x_1 = 2.0979,$ $x_2 = -1.9729,$ $x_3 = 2.0979.$

Remark 3.3. Note that, our results in this paper is extended and generalized results of Elabbasy et al. [6].

Remark 3.4. We can consider many of special cases (not been studied previously) for equation (1.1) as

$$x_{n+1} = \sqrt{\frac{bx_{n-l}}{cx_{n-k} + dx_{n-l}}},$$

$$x_{n+1} = ax_{n-k} + \delta\frac{bx_{n-l}}{cx_{n-k} + dx_{n-l}}.$$

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