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INFLUENCE OF MAGNETIC FIELD ON COUETTE FLOW OF FLUIDS WITH LINEAR DEPENDENCE OF VISCOSITY ON THE PRESSURE

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Abstract

The unsteady simple Couette flow of incompressible viscous fluids with linear dependence of viscosity on the pressure is analytically investigated in presence of a constant magnetic field. Exact expressions are established for steady and transient components of the dimensionless velocity of fluid. Steady velocity is presented in different

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forms whose equivalence is graphically proved. The influence of magnetic field on the fluid behavior is graphically investigated and discussed. It was found that the fluid flows slower and the steady state is earlier reached in the presence of a magnetic field.

1. Introduction

The motion of a fluid between two parallel plates when one of them moves along its plane with a constant velocity V is called the simple Couette flow [1]. This flow is very important both from theoretical and practical point of view having many engineering applications [2]. The first exact solutions corresponding to this flow of incompressible viscous fluids seem to be those of Erdogan [1]. The influence of magnetic field on the unsteady simple Couette flow of same fluids has been relatively recent investigated by Fetecau and Narahari [3] while the first exact solutions for the steady Couette flow of the incompressible viscous fluids with pressure dependent viscosity have been determined by Rajagopal [4]. These solutions have been later extended to the unsteady case by Fetecau and Bridge [5].

The main purpose of this note is to provide the first exact solutions for hydromagnetic unsteady simple Couette flow of the incompressible viscous fluids with linear dependence of viscosity on the pressure whose constitutive equations are given by the relations [4]

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu(p)\boldsymbol{A}; \quad \mu(p) = \alpha p, \quad \alpha > 0, \tag{1}$$

where $\boldsymbol{T}$ is the Cauchy stress, $-p\boldsymbol{I}$ is the reaction stress due to the constraint of incompressibility, $\boldsymbol{A}$ is the first Rivlin-Ericksen tensor, $\mu(p)$ is the fluid viscosity and α is a positive constant. The dimensionless velocity of the fluid, which is presented as a sum of the steady and transient solutions, is used to determine the required time to reach the steady state which is very important for the experimental researchers. The influence of magnetic field on the fluid behavior is graphically depicted and discussed. It was found that the steady state for this flow of the fluids in discussion is earlier obtained in the presence of a magnetic field. Moreover, the fluid flows faster in the absence of the magnetic field.

2. The Problem Presentation

Suppose that an electrical conducting incompressible viscous fluid with linear dependence of viscosity on the pressure is at rest between two infinite horizontal parallel plates. At the moment $t = 0^+$ the lower plate begins to slide along its plane with a constant velocity V and a magnetic field of constant strength B_0 acts perpendicular to plates. We also assume that the fluid is finitely conducting and the magnetic parameter and magnetic Reynolds number are small enough. Consequently, the Hall effects and the induced magnetic field can be neglected [6]. The velocity vector $\boldsymbol{v}$ corresponding to this motion is given by the relation

$$\boldsymbol{v} = \boldsymbol{v}(y, t) = (u(y, t), 0, 0); \quad 0 < y < d, \quad t > 0, \tag{2}$$

in a convenient Cartesian coordinate system x, y and z. Here, d is the distance between plates.

The continuity equation is identically verified. Assuming that the magnetic permeability of the fluid is constant, the imposed electric field is null and the induced magnetic field can be neglected in comparison with the applied magnetic field. The balance of linear momentum reduces to the next two relevant partial differential equations

$$\rho \frac{\partial u(y, t)}{\partial t} = \frac{\partial}{\partial y}\left[\mu(p) \frac{\partial u(y, t)}{\partial y}\right] - \sigma B_0^2 u(y, t); \quad \frac{\partial p(y, t)}{\partial y} + \rho g = 0, \tag{3}$$

in which ρ is the fluid density, σ is its electrical conductivity and g is the acceleration due to gravity. Considering a pressure field independent of time t [4] and integrating the second equation from (3) one finds that

$$p(y) = \rho g(d - y) + p_d, \quad p_d = p(d). \tag{4}$$

From the relations (1), $(3)_1$ and (4) one obtains the governing equation

$$\alpha[\rho g(d-y)+p_d]\frac{\partial^2 u(y,t)}{\partial y^2}-\alpha\rho g\frac{\partial u(y,t)}{\partial y}-\sigma B_0^2 u(y,t)=\rho\frac{\partial u(y,t)}{\partial y};$$

$$0<y<d, \quad t>0, \quad (5)$$

for the fluid velocity $u(y, t)$. The corresponding initial and boundary conditions are

$$u(y,0)=0, \quad 0\le y\le d; \quad u(0,t)=V, \quad u(d,t)=0, \quad t>0. \quad (6)$$

The non-trivial shear stress $\tau(y, t)$, as it results from Eq. (1), is given by the relation [5]

$$\tau(y,t)=\alpha[\rho g(d-y)+p_d]\frac{\partial u(y,t)}{\partial y}; \quad 0<y<d, \quad t>0. \quad (7)$$

Introducing the next non-dimensional variables and function

$$y^*=\frac{1}{d}y, \quad t^*=\frac{\alpha g}{d}t, \quad u^*=\frac{1}{V}u, \quad \tau^*=\frac{1}{\alpha\rho g V}\tau, \quad (8)$$

and dropping out the star notation, one attains to the initial and boundary value problem

$$(1-y+\beta)\frac{\partial^2 u(y,t)}{\partial y^2}-\frac{\partial u(y,t)}{\partial y}-Mu(y,t)=\frac{\partial u(y,t)}{\partial t};$$

$$0<y<1, \quad t>0, \quad (9)$$

$$u(y,0)=0,\ 0\le y\le 1; \quad u(0,t)=1,\ u(1,t)=0,\ t>0, \quad (10)$$

in which the magnetic parameter M and the constant β are defined by the next relations

$$M=\frac{\sigma B_0^2}{\rho}\frac{d}{\alpha g}=\frac{d}{\alpha\rho g}\sigma B_0^2, \quad \beta=\frac{p_d}{\rho g d}. \quad (11)$$

The corresponding shear stress $\tau(y, t)$ can be obtained using the next relation

$$\tau(y, t) = (1 - y + \beta)\frac{\partial u(y, t)}{\partial y}; \quad 0 < y < 1, \quad t > 0. \tag{12}$$

3. Solution

In order to determine the velocity field $u(y, t)$ that satisfies the initial and boundary value problem (9) and (10), we firstly make the change of independent variable $y = 1 + \beta - r^2/4$. One obtains the following partial differential equation

$$\frac{\partial^2 u(r, t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r, t)}{\partial r} - Mu(r, t) = \frac{\partial u(r, t)}{\partial t}; \quad a < r < b, \quad t > 0, \tag{13}$$

with the initial and boundary conditions

$$u(r, 0) = 0, \quad a \le r \le b; \quad u(a, t) = 0, \quad u(b, t) = 1, \quad t > 0, \tag{14}$$

where $a = 2\sqrt{\beta}$ and $b = 2\sqrt{\beta + 1}$.

Now, making the change of unknown function

$$u(r, t) = w(r, t) + \frac{r - a}{b - a}H(t); \quad a < r < b, \quad t \ge 0, \tag{15}$$

where $H(\cdot)$ is the Heaviside unit step function one attains to the partial differential equation

$$\frac{\partial^2 w(r, t)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r, t)}{\partial r} - Mw(r, t) + c(r, t) = \frac{\partial w(r, t)}{\partial t};$$

$$a < r < b, \quad t > 0, \tag{16}$$

where $c(r, t) = \frac{1}{b - a}\left\{\frac{1}{r}H(t) - (r - a)[\delta(t) + MH(t)]\right\}$ and $\delta(t) = H'(t)$ is the Dirac delta function. The new function $w(r, \cdot)$ has to satisfy the initial and boundary conditions

$$w(r, 0) = 0, \quad a \leq r \leq b; \quad w(a, t) = w(b, t) = 0, \quad t > 0. \tag{17}$$

In order to solve the partial differential equation (16) with the initial and boundary conditions (17), the finite Hankel transform and its inverse defined by the relations [7]

$$\begin{cases} w_H(n, t) = \displaystyle\int_a^b rw(r, t)B(r, r_n)dr, \\ w(r, t) = \dfrac{\pi^2}{2}\displaystyle\sum_{n=1}^{\infty} \dfrac{r_n^2 J_0^2(br_n)B(r, r_n)}{J_0^2(ar_n) - J_0^2(br_n)} w_H(n, t), \end{cases} \tag{18}$$

will be used. In above relations $w_H(n, t)$ is the finite Hankel transform of $w(r, t)$, r_n are the positive roots of the transcendental equation $B(b, r) = 0$ in which

$$B(r, r_n) = Y_0(ar_n)J_0(rr_n) - J_0(ar_n)Y_0(rr_n), \quad n = 1, 2, 3, \ldots \tag{19}$$

and $J_0(\cdot)$ and $Y_0(\cdot)$ are standard Bessel functions of first and second kind and zero order.

Consequently, multiplying Eq. (16) by $rB(r, r_n)$, integrating the result from a to b and bearing in mind the initial and boundary conditions (17) and the known result [7]

$$\int_a^b rB(r, r_n)\left[\frac{\partial^2 w(r, t)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r, t)}{\partial r}\right]dr$$

$$= \frac{2}{\pi}\left[w(b, t)\frac{J_0(ar_n)}{J_0(br_n)} - w(a, t)\right] - r_n^2 w_H(n, t), \tag{20}$$

one finds that $w_H(n, t)$ has to satisfy the boundary value problem

$$\frac{\partial w_H(n, t)}{\partial t} + (r_n^2 + M)w_H(n, t) = d_n(t),$$

$$w_H(n, 0) = 0; \quad t > 0, \quad n = 1, 2, 3, ..., \tag{21}$$

where $d_n(t) = \frac{1}{b-a}\{\alpha_n H(t) - \beta_n[\delta(t) + MH(t)]\}$ and

$$\alpha_n = \int_a^b B(r, r_n)dr, \quad \beta_n = \int_a^b r(r-a)B(r, r_n)dr. \tag{22}$$

Solving the ordinary differential equation (21), introducing the result in the equality $(18)_2$ and coming back to the original variable and function, the dimensionless velocity field $u(y, t)$ can be written as sum of steady and transient components, i.e., $u(y, t) = [u_s(y) + u_t(y, t)]H(t)$. The steady and transient velocities $u_s(y)$ and $u_t(y, t)$, respectively, have the expressions

$$u_s(y) = \frac{\sqrt{1-y+\beta} - \sqrt{\beta}}{\sqrt{1+\beta} - \sqrt{\beta}}$$

$$+ \frac{\pi^2}{2(b-a)} \sum_{n=1}^{\infty} \frac{r_n^2 J_0^2(br_n) B\left(2\sqrt{1-y+\beta}, r_n\right)}{J_0^2(ar_n) - J_0^2(br_n)} \frac{\alpha_n - \beta_n M}{r_n^2 + M}, \tag{23}$$

$$u_t(y, t) = -\frac{\pi^2}{2(b-a)}$$

$$\times \sum_{n=1}^{\infty} \frac{r_n^2 J_0^2(br_n) B\left(2\sqrt{1-y+\beta}, r_n\right)}{J_0^2(ar_n) - J_0^2(br_n)} \frac{\beta_n r_n^2 + \alpha_n}{r_n^2 + M} e^{-(r_n^2+M)t}. \tag{24}$$

Simple computations show that the dimensionless velocity $u(y, t)$ satisfies all imposed initial and boundary conditions. Moreover, making $M = 0$ in above relations the solutions obtained by Fetecau and Bridges [5] are recovered.

The corresponding shear stresses are given by the following relations

$$\tau_s(y) = -\frac{\sqrt{1-y+\beta}}{2(\sqrt{1+\beta}-\sqrt{\beta})}$$

$$+\frac{\pi^2\sqrt{1-y+\beta}}{2(b-a)}\sum_{n=1}^{\infty}\frac{r_n^3 J_0^2(br_n)C\left(2\sqrt{1-y+\beta},\ r_n\right)}{J_0^2(ar_n)-J_0^2(br_n)}\frac{\alpha_n-\beta_n M}{r_n^2+M}, \quad (25)$$

$$\tau_t(y,\ t) = -\frac{\pi^2\sqrt{1-y+\beta}}{2(b-a)}$$

$$\times\sum_{n=1}^{\infty}\frac{r_n^3 J_0^2(br_n)C\left(2\sqrt{1-y+\beta},\ r_n\right)}{J_0^2(ar_n)-J_0^2(br_n)}\frac{\beta_n r_n^2+\alpha_n}{r_n^2+M}e^{-(r_n^2+M)t}, \quad (26)$$

in which

$$C\left(2\sqrt{1-y+\beta},\ r_n\right) = Y_0(ar_n)J_1(2r_n\sqrt{1-y+\beta}) - J_0(ar_n)Y_1(2r_n\sqrt{1-y+\beta});$$

$n = 1, 2, 3, \ldots$ and $J_1(\cdot)$ and $Y_1(\cdot)$ are standard Bessel functions of the first and second kind of one order.

An equivalent form for the dimensionless steady velocity $u_s(y)$, namely

$$u_s(y) = \frac{K_0(2\sqrt{BM})I_0(2\sqrt{(1-y+\beta)M}) - I_0(2\sqrt{BM})K_0(2\sqrt{(1-y+\beta)M})}{K_0(2\sqrt{BM})I_0(2\sqrt{(1+\beta)M}) - I_0(2\sqrt{BM})K_0(2\sqrt{(1+\beta)M})};$$

$$M \neq 0, \quad (27)$$

has been determined solving the corresponding boundary value problem

$$\frac{d^2u_s(r)}{dr^2}+\frac{1}{r}\frac{du_s(r)}{dr}-Mu_s(r) = 0; \quad u(a) = 0, \quad u(b) = 1. \quad (28)$$

$I_0(\cdot)$ and $K_0(\cdot)$ from Eq. (27) are modified Bessel functions of first and second kind and zero order. Indeed, making the change of variable

$r = q / \sqrt{M}$ one obtains the Bessel equation

$$\frac{d^2 u_s(q)}{dq^2} + \frac{1}{r}\frac{du_s(q)}{dq} - u_s(q) = 0; \quad u(a\sqrt{M}) = 0, \quad u(b\sqrt{M}) = 1. \quad (29)$$

Solving this boundary value problem and coming back to the initial variable one obtains the relation (27). The steady solution $u_s(y)$ is independent of initial condition but it satisfies the governing equation and boundary conditions. The equivalence of the expressions of $u_s(y)$ given by the relations (23) and (27) has been graphically proved by means of Figure 1.

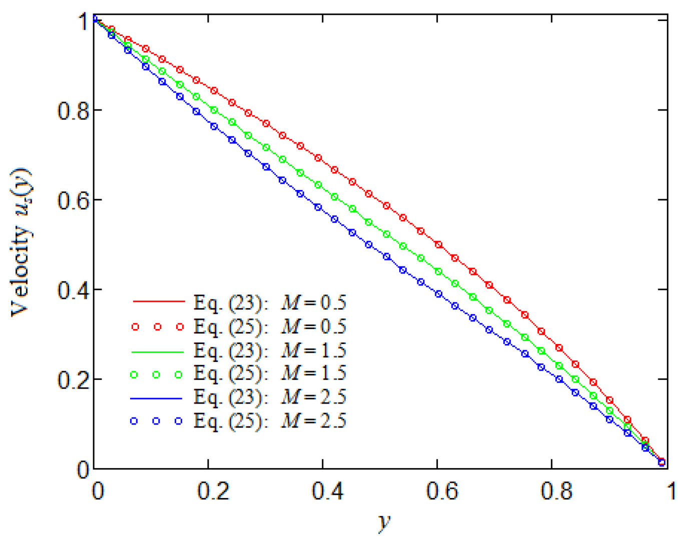

Figure 1. Equivalence of the expressions of $u_s(y)$ given by Eqs. (23) and (25), for $\beta = 0.5$ and three values of the magnetic parameter M.

This figure clearly shows that $u_s(y)$ is a decreasing function with respect to the magnetic parameter M. Consequently, the fluid flows slower in the presence of a magnetic field.

An equivalent form for the dimensionless shear stress $\tau_s(y)$, namely

$$\tau_s(y) = (\sqrt{(1-y+\beta)M})$$

$$\times \frac{K_0(2\sqrt{BM})I_0'(2\sqrt{(1-y+\beta)M}) + I_0(2\sqrt{BM})K_0'(2\sqrt{(1-y+\beta)M})}{I_0(2\sqrt{BM})K_0(2\sqrt{(1+\beta)M}) - K_0(2\sqrt{BM})I_0(2\sqrt{(1+\beta)M})};$$

$$M \neq 0, \tag{30}$$

has been obtained introducing $u_s(y)$ from Eq. (27) in (12).

4. Some Numerical Results and Conclusions

In this short note the unsteady simple Couette flow of incompressible viscous fluids with linear dependence of viscosity on the pressure has been analytically investigated. Analytic expressions are determined for the steady and transient components of the dimensionless starting velocity and non-trivial shear stress fields $u(y, t)$ and $\tau(y, t)$, respectively. As expected, in the absence of magnetic field these expressions reduce to known results from the literature. For the results validation, the steady velocity field $u_s(y)$ is presented in two different forms whose equivalence was graphically proved.

This unsteady flow, whose theoretical and practical importance is well known in the literature, becomes steady or permanent in time. In practice, a very important problem for such flows is to know the required time to touch the steady state. From mathematical point of view, this is the time after which the diagrams of starting solution $u(y, t)$ overlap with that of its steady component $u_s(y)$. From Figure 2 it results that the required time to reach the steady state decreases for increasing values of the magnetic parameter M. Consequently, the steady state for hydromagnetic unsteady simple Couette flow of incompressible viscous fluids with linear dependence of viscosity on the pressure as well as in the case of ordinary viscous fluids is earlier obtained in the presence of a magnetic field. Furthermore, as it was to be expected, from Figure 2 it also results that the fluid velocity is an increasing function with regard to the time t. In conclusion, the main outcomes that have been here obtained are:

- Exact solutions were established for the hydromagnetic unsteady simple Couette flow of incompressible viscous fluids with linear dependence of viscosity on the pressure.

- Their correctness has been graphically proved and it was shown that the fluid flows slower and the steady state is rather touched in the presence of a magnetic field.

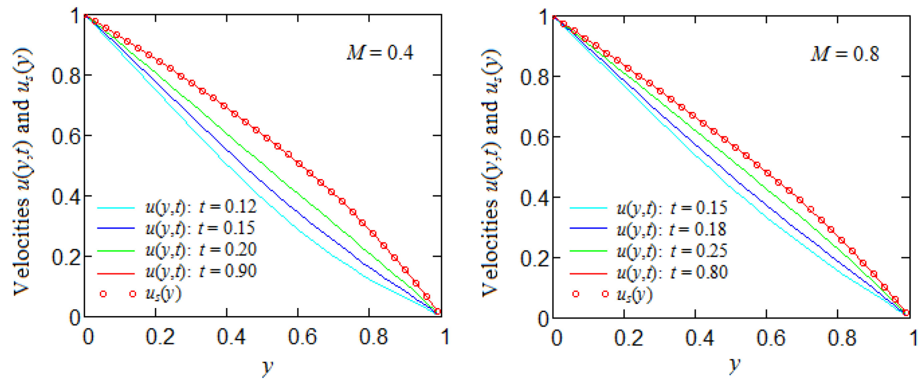

Figure 2. Convergence of the starting velocity $u(y, t)$ to its steady component $u_s(y)$ when $\beta = 0.5$, $M = 0.4$ or $M = 0.8$ and increasing values of t.

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