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HERTZ AND WHITTAKER SCALAR POTENTIALS

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Abstract

We prove that the scalar Whittaker potentials are a particular case of the scalar Hertz potentials.

1. Microscopic Electrodynamics

It has been known to Hertz [1, 2] that the microscopic electrodynamics defined by one of the two couples $(\mathbf{e}, \mathbf{b})$, electric field and magnetic induction or $(\mathbf{h}, \mathbf{d})$ magnetic field and electric displacement depends, in absence of charges and currents, on two scalar potentials $\prod$, Ω solutions of the wave equation $(\Delta - c^{-2}\partial_t^{-2})\{\prod, \Omega\} = 0$, and [2]

$$\mathbf{e} = \nabla\nabla.(\mathbf{g}\prod) - c^{-2}\partial_t^{-2}(\mathbf{g}\prod) - c^{-1}\partial_t(\nabla \wedge \mathbf{g}\Omega),$$

$$\mathbf{b} = c^{-1}\partial_t(\nabla \wedge \mathbf{g}\Pi) + \nabla \wedge \nabla \wedge \mathbf{g}\Omega \tag{1}$$

in which $\mathbf{g}$ is an arbitrary unit vector.

Some years later, Whittaker [3], made a similar remark and, in terms of two scalar potentials, F, G solutions of the wave equation, he obtained for $\{\mathbf{e}, \mathbf{b}\}$ the following expressions (in fact for the couple $\{\mathbf{h}, \mathbf{d}\}$)

$$e_x = \partial_x\partial_z F + c^{-1}\partial_y\partial_t G, \quad e_y = \partial_y\partial_z F - c^{-1}\partial_x\partial_t G, \quad e_z = \partial_z{}^2 F - c^{-2}\partial_t{}^2 F,$$

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$$b_x = -\partial_x \partial_z G + c^{-1} \partial_y \partial_t F, \;\; b_y = -\partial_y \partial_z G - c^{-1} \partial_x \partial_t F, \;\; b_z = -\partial_z{}^2 G + c^{-2} \partial_t{}^2 G. \quad (2)$$

Now, taking for the vector **g** the unit vector $\mathbf{i}_z$ in the z direction, so that $g_x = g_y = 0$, $g_z = 1$, it is easy to check that the Hertz expressions (1) reduce to (2) with $F = \prod$, $G = -\Omega$. So, the Whittaker potentials are a particular case of Hertz potentials.

2. Macroscopic Electrodynamics

For the macroscopic electrodynamics [4] requiring the four fields **E**, **B**, **H**, **D**, where in terms of electric polarization **P** and magnetization **M** [3]

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \qquad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} \tag{3}$$

the previous results generalize and, still in absence of charges and currents, the scalar Hertz potentials being solutions, in a isotropic homogeneous medium with refractive index n, of the wave equation $(\Delta - n^2 c^{-2} \partial_t{}^2)\{\prod, \Omega\} = 0$, it comes [2]

$$\mathbf{E} = \nabla\nabla \cdot (\mathbf{g} \prod) - n^2 c^{-2} \partial_t{}^{-2} (\mathbf{g} \prod) - n^2 c^{-1} \partial_t (\nabla \wedge \mathbf{g}\Omega),$$

$$\mathbf{B} = n^2 c^{-1} \partial_t (\nabla \wedge \mathbf{g} \prod) + n^2 \nabla \wedge \nabla \wedge \mathbf{g}\Omega \tag{4}$$

and [1]

$$\mathbf{H} = \nabla\nabla \cdot (\mathbf{g}\Omega) + n^2 c^{-2} \partial_t{}^{-2} (\mathbf{g}\Omega) + n^2 c^{-1} \partial_t (\nabla \wedge \mathbf{g} \prod),$$

$$\mathbf{D} = n^2 c^{-1} \partial_t (\nabla \wedge \mathbf{g}\Omega) + n^2 \nabla \wedge \nabla \wedge \mathbf{g} \prod. \tag{5}$$

Substituting (4) and (5) into (3) gives the vectors **P**, **M**.

Taking for **g** the unit vector $\mathbf{i}_z$ and changing $(\prod, \Omega)$ into $(F, -G)$ generalizes the Whittaker potentials to the macroscopic electrodynamics.

Remark 1. The potential vector **A** and scalar ϕ are defined in terms of Hertz potentials by the relations [1], leaving aside an arbitrary scalar function Ψ

$$\mathbf{A} = n^2 c^{-1} \partial_t (\mathbf{g} \prod) + n^2 \nabla \wedge \mathbf{g}\Omega, \qquad \phi = -\nabla \cdot \mathbf{g} \prod. \tag{6}$$

Remark 2. In many situations, it is interesting to work with complex fields since together with potentials they take a compact form. For instance, with

$$\mathbf{Q} = \mathbf{B} + in\mathbf{E} \tag{7}$$

it comes [1]

$$\mathbf{Q} = n^2 c^{-1} \partial_t \nabla \wedge \Gamma + in \nabla \wedge \nabla \wedge \Gamma \tag{8}$$

with $\Gamma = -i(\mu/\varepsilon)^{1/2}\, \mathbf{g}\Omega.$

Of course all these results hold valid for Whittaker potentials.

3. Discussion

Whittaker potentials and more generally the use of scalar potentials in electromagnetism, have recently known some revival [5, 6, 7, 8]. Reference to Hertz potentials would have been justified.

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