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FRENET-SERRET FORMULAS OF *q*-CURVES

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Abstract

This paper explores the effect of q-derivative on curves in three dimensional Euclidean space and constructs a q-frame $\{T_q, N_q, B_q\}$ with the help of the Frenet frame field $\{T, N, B\}$ at any point.

1. Introduction

To analyze a curve, Frenet-Serret frame is one of the most important part in Differential Geometry of curves. All detailed information about our main subject was introduced in [1] and [2]. Another subject that has a wide range applications and published about them is the Quantum Calculus recently conducted by [3]. In this

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context, we see one of the different approach in [4, 6]. Also we utilize some functional relations in [5].

In this research, we foresee that there is a close relationship between quantum behaviour and curves in differential geometry. And we will consider the derivational relations which hold on important place in the study of the process as a starting point by the q-derivative approach.

2. Frenet-Serret Formulas

Let $\alpha : I \subset \mathbb{R} \to E^3,\ t \mapsto \alpha(t) = (\alpha_1(t),\ \alpha_2(t),\ \alpha_3(t))$ be a regular curve in 3-dimensional Euclidean space. The tangent, normal and binormal unit vectors construct a frame on curve at any point. The three vectors T, N, and B are called Frenet-Serret vectors and the triple $\{T,\ N,\ B\}$ is called the Frenet frame field. They together form an orthonormal basis which span $\mathbb{R}^3$.

The Frenet-Serret vectors hold:

$$T = \frac{\alpha'(t)}{|\alpha'(t)|},$$

$$B = \frac{\alpha'(t) \times \alpha''(t)}{\|\alpha'(t) \times \alpha''(t)\|},$$

$$N = B \times T.$$

κ is the curvature of a curve and measures how much the curve deviates from a straight line, τ is the torsion of a curve and measures how much the curve deviates from the osculating plane.

$$\kappa = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3},$$

$$\tau = \frac{\langle \alpha'(t) \times \alpha''(t),\ \alpha'''(t) \rangle}{\|\alpha'(t) \times \alpha''(t)\|^2}. \ [2]$$

3. q-Derivative

In Quantum Calculus the q-derivative of function $f(x)$ is defined as

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{1}$$

Here, the definition reduces to the ordinary derivative when the limit $q \to 1$. It is clear that as with the ordinary derivative taking the q-derivative of a function is a linear operator which means $D_q(af(x) + bg(x)) = aD_q f(x) + bD_q g(x)$ for any constants a and b, [3, 4].

We can write the second q-derivative $D_q^2 f(x)$ and the third q-derivative $D_q^3 f(x)$ with the help of the formula of q-derivative:

$$D_q^2 f(x) = \frac{f(q^2 x) - (1+q) f(qx) + qf(x)}{q(q-1)^2 x^2},$$

$$D_q^3 f(x) = \frac{f(q^3 x) - (1+q+q^2) f(q^2 x) + (1+q+q^2) qf(qx) - q^3 f(x)}{q^3 (q-1)^3 x^3}. \tag{2}$$

Together with these definitions, we will use the Euclidean norm and the Euclidean vectoral product definitions.

4. Frenet-Serret Formulas with using q-Derivative

Suppose $\alpha(t)$ is a curve with t parameter in three dimensional Euclidean space. The q-derivative of $\alpha(t)$ is

$$D_q \alpha(t) = \frac{\alpha(qt) - \alpha(t)}{(q-1)t},$$

and the second q-derivative is

$$D_q^2\alpha(t) = \frac{\alpha(q^2t) - (1+q)\alpha(qt) + q\alpha(t)}{q(q-1)^2t^2},$$

and the third q-derivative is

$$D_q^3\alpha(t) = \frac{\alpha(q^3t) - (1+q+q^2)\alpha(q^2t) + (1+q+q^2)q\alpha(qt) - q^3\alpha(t)}{q^3(q-1)^3t^3}.$$

From now on, we will call the whole frame vectors with q-derivative as q-frame. We can write as a q-tangent vector field,

$$T_q = \frac{D_q\alpha(t)}{\|D_q\alpha(t)\|}. \tag{3}$$

Then, we can determine, $\langle D_q\alpha(t), D_q\alpha(t) \times D_q^2\alpha(t)\rangle = 0,$ and with the help of this equality we can find vector B_q perpendicular to vector T_q. And the definition of the q-binormal vector perpendicular to T_q is

$$B_q = \frac{D_q\alpha(t) \times D_q^2\alpha(t)}{\|D_q\alpha(t) \times D_q^2\alpha(t)\|}. \tag{4}$$

With using the Equations (1) and (2), we find

$$D_q\alpha(t) \times D_q^2\alpha(t) = \frac{\alpha(qt) \times \alpha(q^2t) + \alpha(q^2t) \times \alpha(t) + \alpha(t) \times \alpha(qt)}{q(q-1)^3t^3}. \tag{5}$$

With the norm definition we can use

$$\|D_q\alpha(t)\| = \frac{\|\alpha(qt) - \alpha(t)\|}{(q-1)t}. \tag{6}$$

We will now apply the formula of N given in Section 2, we use the Equations (3), (4), (5) and (6) to find N_q.

$$N_q = B_q \times T_q$$

$$= \frac{D_q^2\alpha(t)\langle D_q\alpha(t),\, D_q\alpha(t)\rangle - D_q\alpha(t)\langle D_q\alpha(t),\, D_q^2\alpha(t)\rangle}{\|D_q\alpha(t)\times D_q^2\alpha(t)\|\,\|D_q\alpha(t)\|}$$

$$= \frac{[\alpha(qt)-\alpha(q^2t)]\langle \alpha(t),\, \alpha(qt)-\alpha(t)\rangle + [\alpha(q^2t)-\alpha(qt)]\langle \alpha(qt),\, \alpha(qt)-\alpha(t)\rangle}{\|\alpha(qt)\times\alpha(q^2t)+\alpha(q^2t)\times\alpha(t)+\alpha(t)\times\alpha(qt)\|\,\|\alpha(qt)-\alpha(t)\|}$$

$$+ \frac{[\alpha(t)-\alpha(qt)]\langle \alpha(q^2t),\, \alpha(qt)-\alpha(t)\rangle}{\|\alpha(qt)\times\alpha(q^2t)+\alpha(q^2t)\times\alpha(t)+\alpha(t)\times\alpha(qt)\|\,\|\alpha(qt)-\alpha(t)\|}.$$

The q-curvature κ_q and the q-torsion τ_q is found with the aid of the formulas in Section 2:

$$\kappa_q = \frac{\|D_q\alpha(t)\times D_q^2\alpha(t)\|}{\|D_q\alpha(t)\|^3}$$

$$= \frac{1}{q}\frac{\|\alpha(qt)\times\alpha(q^2t)+\alpha(q^2t)\times\alpha(t)+\alpha(t)\times\alpha(qt)\|}{\|\alpha(qt)-\alpha(t)\|^3},$$

and

$$\tau_q = \frac{\langle D_q\alpha(t)\times D_q^2\alpha(t),\, D_q^3\alpha(t)\rangle}{\|D_q\alpha(t)\times D_q^2\alpha(t)\|^2}$$

$$= \frac{1}{q^2}\frac{\langle \alpha(qt)\times\alpha(q^2t)+\alpha(q^2t)\times\alpha(t)+\alpha(t)\times\alpha(qt),\, \alpha(q^3t)\rangle - \langle \alpha(t)\times\alpha(qt),\, \alpha(q^2t)\rangle}{\|\alpha(qt)\times\alpha(q^2t)+\alpha(q^2t)\times\alpha(t)+\alpha(t)\times\alpha(qt)\|^2}.$$

If we need to demonstrate what we have achieved with an Example 1, calculate q-frame, q-curvature and q-torsion for the curve $\alpha(t) = (\frac{3}{5}\cos(t),\, \sin(t),\, \frac{4}{5}\cos(t))$.

The q-derivative of the given curve is

$$D_q\alpha(t) = \frac{1}{(q-1)t}\left[\frac{3}{5}(\cos(qt)-\cos(t)),\, \sin(qt)-\sin(t),\, \frac{4}{5}(\cos(qt)-\cos(t))\right],$$

and the norm of $D_q\alpha(t)$ is

$$\|D_q\alpha(t)\| = \frac{1}{(q-1)t}\sqrt{2-2\cos(t-qt)}.$$

This curve is a unit velocity curve in ordinary differential geometry however, in terms of Euclidean meaning, when quantum derivative is used, curve is not always unit velocity. Then we have

$$D_q^2\alpha(t) = \frac{1}{q(q-1)^2t^2}\Big[\frac{3}{5}(\cos(q^2t)-(1+q)\cos(qt)+q\cos(t)),\ \sin(q^2t)$$

$$-(1+q)\sin(qt)+q\sin(t),\ \frac{4}{5}(\cos(q^2t)-(1+q)\cos(qt)+q\cos(t))\Big],$$

$$D_q^3\alpha(t) = \frac{1}{q^3(q-1)^3t^3}\Big[\frac{3}{5}(\cos(q^3t)-(1+q+q^2)\cos(q^2t)$$

$$+(1+q+q^2)q\cos(qt)-q^3\cos(t)),\ \sin(q^3t)-(1+q+q^2)\sin(q^2t)$$

$$+(1+q+q^2)q\sin(qt)-q^3\sin(t),\ \frac{4}{5}(\cos(q^3t)-(1+q+q^2)\cos(q^2t)$$

$$+(1+q+q^2)q\cos(qt)-q^3\cos(t))\Big],$$

$$D_q\alpha(t)\times D_q^2\alpha(t) = \frac{1}{q(q-1)^3t^3}\Big[\frac{4}{5}\sin(qt-q^2t)+\frac{4}{5}\sin(t-qt)+\frac{4}{5}\sin(q^2t-t),\ 0,$$

$$-\frac{3}{5}\sin(qt-q^2t)-\frac{3}{5}\sin(t-qt)-\frac{3}{5}\sin(q^2t-t)\Big],$$

thus

$$\langle D_q\alpha(t)\times D_q^2\alpha(t),\ D_q^3\alpha(t)\rangle = 0,$$

and

$$\|D_q\alpha(t)\times D_q^2\alpha(t)\| = \frac{1}{q(q-1)^3t^3}\,[\,\sin(qt-q^2t)+\sin(t-qt)+\sin(q^2t-t)].$$

We calculate the derivatives and get the value of inner product equal zero, it helps to write the vectors B_q and N_q.

$$T_q = \frac{D_q\alpha(t)}{\|D_q\alpha(t)\|}$$

$$= \left[\frac{3}{5}\left(\frac{\cos(qt)-\cos(t)}{\sqrt{2-2\cos(t-qt)}}\right), \frac{\sin(qt)-\sin(t)}{\sqrt{2-2\cos(t-qt)}}, \frac{4}{5}\left(\frac{\cos(qt)-\cos(t)}{\sqrt{2-2\cos(t-qt)}}\right)\right],$$

$$B_q = \frac{D_q\alpha(t)\times D_q^2\alpha(t)}{\|D_q\alpha(t)\times D_q^2\alpha(t)\|} = \left(\frac{4}{5}, 0, -\frac{3}{5}\right),$$

$$N_q = \left[\frac{3}{5}\left(\frac{\sin(qt)-\sin(t)}{\sqrt{2-2\cos(t-qt)}}\right), -\frac{\cos(qt)-\cos(t)}{\sqrt{2-2\cos(t-qt)}}, \frac{4}{5}\left(\frac{\sin(qt)-\sin(t)}{\sqrt{2-2\cos(t-qt)}}\right)\right].$$

We calculate q-curvature and q-torsion for the curve $\alpha(t)$, here it is clear that, $\tau_q = 0$ equality shows the curve $\alpha(t)$ is planar same as in ordinary differential geometry in E^3 :

$$\kappa_q = \frac{\|D_q\alpha(t)\times D_q^2\alpha(t)\|}{\|D_q\alpha(t)\|^3} = \left(\frac{1}{q}\right)\frac{\|\sin(qt-q^2t)+\sin(t-qt)+\sin(q^2t-t)\|}{[\sqrt{2-2\cos(t-qt)}\,]^3},$$

and

$$\tau_q = \frac{\langle D_q\alpha(t)\times D_q^2\alpha(t),\ D_q^3\alpha(t)\rangle}{\|D_q\alpha(t)\times D_q^2\alpha(t)\|^2} = 0.$$

Now look at what we have obtain by entering values for q and parameter t on Example 1. When $q = 0.3$ and $t = \frac{\pi}{2}$ gives:

$T_q = (0.51, -0.52, 0.68)$, $B_q = (0.8, 0, -0.6)$, $N_q = (-0.31, -0.85, -0.42)$, (Figure 1).

When $q = 0.5$ and $t = \frac{\pi}{2}$ gives:

$T_q = (0.55, -0.38, 0.74)$, $B_q = (0.8, 0, -0.6)$, $N_q = (-0.23, -0.92, -0.31)$, (Figure 1).

When $q = 0.7$ and $t = \frac{\pi}{2}$ gives:

$T_q = (0.58, -0.23, 0.78)$, $B_q = (0.8, 0, -0.6)$, $N_q = (-0.14, -0.97, -0.19)$, (Figure 1).

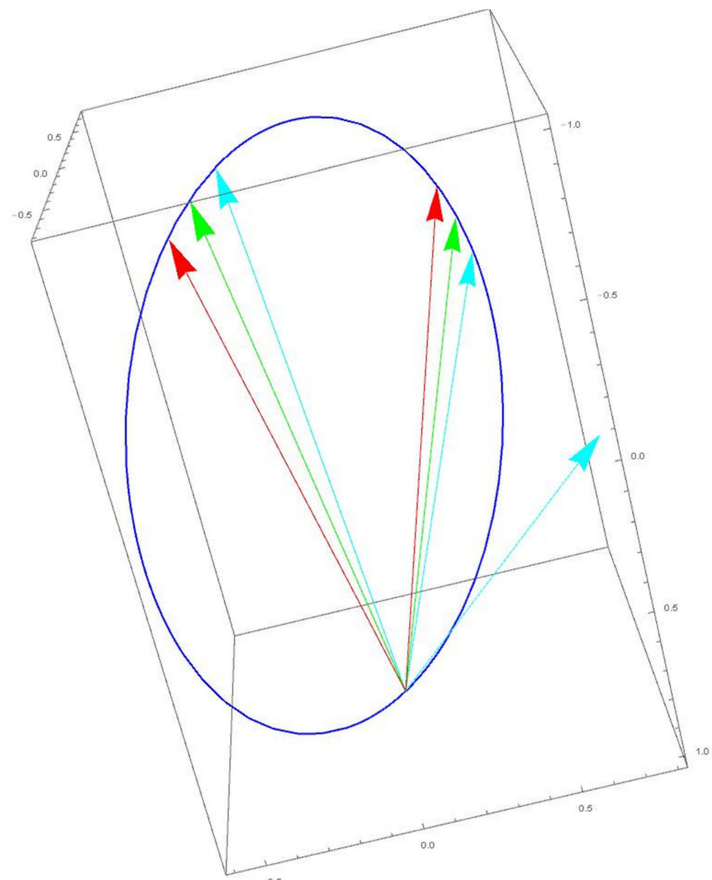

Figure 1. $\{T_q, N_q, B_q\}$ frame at $q = 0.3$, $q = 0.5$, $q = 0.7$ for Example 1.

Another example which is not a unit velocity curve in E^3, in Example 2, we analyze the q-frame on $\alpha(t) = (4\cos(t), 4\sin(t), 3t)$.

When $q = 0.1$ and $t = 3$ gives:

$T_q = (0.69, 0.05, -0.72)$, $B_q = (0.29, 0.16, 0.27)$, $N_q = (-0.1, -0.4, -0.12)$, (Figure 2).

When $q = 0.5$ and $t = 3$ gives:

$T_q = (0.6, 0.48, -0.64)$, $B_q = (0.36, 0.06, 0.38)$, $N_q = (0.23, -0.46, -0.13)$, (Figure 2).

When $q = 0.9$ and $t = 3$ gives:

$T_q = (0.23, 0.76, -0.6)$, $B_q = (0.04, -0.08, 0.11)$, $N_q = (0.13, -0.05, -0.01)$, (Figure 2).

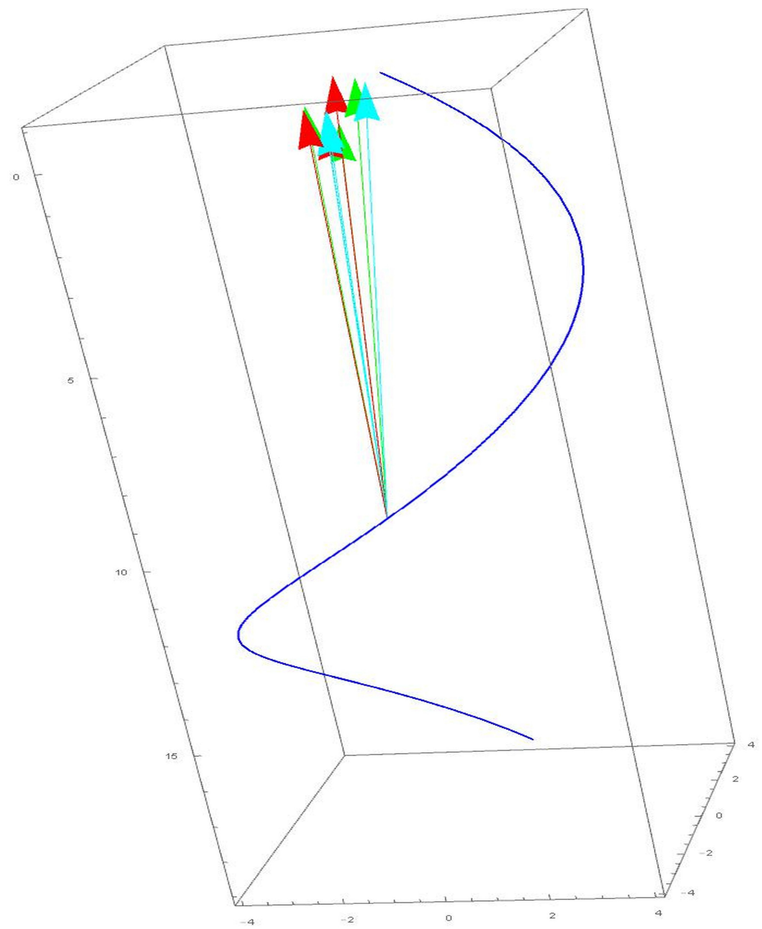

Figure 2. $\{T_q, N_q, B_q\}$ frame at $q = 0.1$, $q = 0.5$, $q = 0.9$ for Example 2.

5. Conclusion

In derivative quotations, there may be a chord for each value of $0 < q < 1$. A tangent vector can be drawn in every point which is parallel to these chords with some value of an angle. The curve which is called as envelope curve that has been created by those tangent vectors corresponds to the curve which is mentioned at the beginning. In future work, we can also examine this surface properties.

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