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ESTIMATION MODELS USING MATHEMATICAL CONCEPTS AND NEWTON'S LAWS FOR CONIC SECTION TRAJECTORIES ON EARTH'S SURFACE

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Abstract

Suppose an object is thrown obliquely near the earth's surface. Theoretically, its trajectory is a parabolic. We have known that the combinations of Uniform Linear Motion (ULM) with constant velocity on *X*-axis and Uniformly Accelerated Linear Motion (UALM) on *Y*-axis produce a parabolic motion. Based on mathematical concepts and Newton's Laws of motion, a semicircular motion also can be considered as the combinations of different motion on *X*-axis and *Y*-axis. This theoretical study attempted to compare parabolic motion and semicircular motion. We have proved that a semicircular motion on earth's surface with small radius is a particular case of parabolic motion with different coefficients of highest order at time *t*.

1. Introduction

This article compares between parabolic motion and semicircular motion. We will also investigate equations on *X*-axis and *Y*-axis using mathematical concepts and Newton's law of motion. Theoretically in physics, if we throw an object obliquely near the earth's surface, the trajectory will be a parabolic. What if the trajectory from an object thrown obliquely near the earth's surface is a semicircular? What

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combinations on *X*-axis and *Y*-axis will be? This article will combine mathematical concepts like parabolic and circle equations, and then compare with the equations which are derived from Newton's law of motion.

2. Theoretical Review

Before we discuss about semicircular motion, we recall the parabolic motion as follows:

2.1. Parabolic motion

Suppose an object is thrown obliquely near the earth's surface with initial velocity v_0 and α is an angle between initial velocity vector and *X*-axis

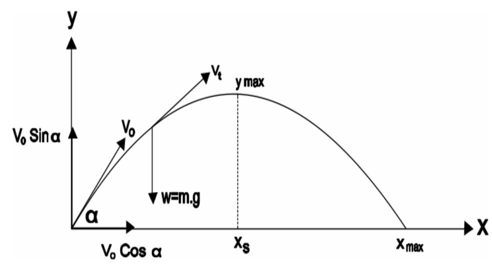

Figure 1. Parabolic trajectory.

In Physics, Figure 1 above can be expressed into equations on *X*-axis and *Y*-axis as follows:

Based on Newton's first law, $\sum F_x = 0$ for constant mass. The horizontal components of *X*-axis are ULM

$$x = v_0 \cos \alpha \, t, \tag{1}$$

$$v_x = v_0 \cos \alpha. \tag{2}$$

Based on Newton's second law, $\sum F_y = ma_y$ for constant mass, the vertical components of *Y*-axis are UALM.

$$y = v_0 \sin \alpha \, t - 0.5gt^2, \tag{3}$$

$$v_y = v_0 \sin \alpha - gt, \tag{4}$$

$$a_y = -g. \tag{5}$$

Substituting (1) into (3) we have:

$$y = v_0 \sin\alpha\left(\frac{x}{v_0 \cos\alpha}\right) - .05g\left(\frac{x}{v_0 \cos\alpha}\right)^2 \tag{6}$$

$$= (\tan\alpha)x - \left(\frac{g}{2{v_0}^2 \cos^2\alpha}\right)x^2, \tag{7}$$

$$y = ax - bx^2. \tag{8}$$

Equation (8) also can be considered as quadratic function in mathematics, we can solve it using mathematical concept as follows:

To get maximum distance $(x_{\max})$, we solve quadratic function above by taking $y = 0$.

$$0 = ax - bx^2,$$

$$0 = (a - bx),$$

$$x_1 = 0 \text{ or } x_2 = \frac{a}{b} = \frac{{v_0}^2 \sin 2\alpha}{g},$$

$$x_2 = x_{\max} = \frac{{v_0}^2 \sin 2\alpha}{g}, \tag{9}$$

$$x_s = \frac{x_1 + x_2}{2} = \frac{{v_0}^2 \sin 2\alpha}{2g} = 0.5x_{\max}. \tag{10}$$

Substituting (10) into (8), we have maximum height:

$$y_{\max} = a\left(\frac{{v_0}^2 \sin 2\alpha}{2g}\right) - b\left(\frac{{v_0}^2 \sin 2\alpha}{2g}\right)^2$$

$$= \tan\alpha\left(\frac{{v_0}^2 \sin 2\alpha}{2g}\right) - \frac{g}{2{v_0}^2 \cos^2\alpha}\left(\frac{{v_0}^2 \sin 2\alpha}{2g}\right)^2$$

$$= \frac{{v_0}^2 \sin^2\alpha}{g} - \frac{{v_0}^2 \sin^2\alpha}{2g}$$

$$= \frac{{v_0}^2 \sin^2\alpha}{2g}. \tag{11}$$

Now, we will attempt to derive equations on *Y*-axis from quadratic function (parabolic equation) and Newton's first law of motion. We recall quadratic function in mathematics, if it passes through maximum or minimum point and one point arbitrary, we have equations as follows:

$$y = c(x - x_p)^2 + y_p, \tag{12}$$

$$y = c(x - x_s)^2 + y_{\max}, \tag{13}$$

$$y = c(x^2 - 2xx_s + x_s^2) + y_{\max},$$

$$y = cx^2 - 2cxx_s + cx_s^2 + y_{\max}. \tag{14}$$

From (13),

$$c = \frac{y - y_{\max}}{(x - x_s)^2}.$$

Since the trajectory passes through the center coordinate, we have

$$c = \frac{-y_{\max}}{(x_s)^2} = \frac{-\tan\theta}{x_s}. \tag{15}$$

Substituting (15) into (14), we have

$$y = \frac{-\tan\theta}{x_s}x^2 + (2\tan\theta)x. \tag{16}$$

Equation (16) corresponds to (8)

$$2\tan\theta = \tan\alpha \quad \text{and} \quad \frac{\tan\theta}{x_s} = \frac{g}{2v_0^2\cos^2\alpha}. \tag{17}$$

If we substitute (1) into (16), we will get equations which corresponds to (3), (4) and (5) by deriving (16) with respect to time *t*.

2.2. Semicircular motion

Suppose an object is thrown obliquely near the earth's surface with initial velocity v_0 and α is an angle between initial velocity vector and *X*-axis. We assume that horizontal *X*-axis is earth's surface for small radius.

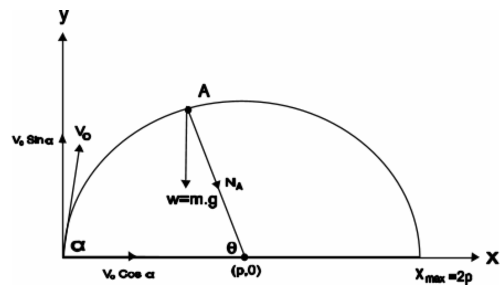

Figure 2. Semicircular trajectory.

From Figure 2, there is fundamental discrepancy between parabolic motion and semicircular motion. If parabolic motion only has an acceleration of gravity on downward direction, while semicircular motion has an acceleration of gravity on downward direction as well as centripetal acceleration that leads to the center (ignoring air friction), we will solve this problem using circle equation as follows:

$$(x - p)^2 + (y - q)^2 = r^2. \tag{18}$$

It is known that the center of the circle is $(p, 0)$, so (18) becomes

$$(x - p)^2 + y^2 = r^2, \tag{19}$$

$$x^2 - 2px + p^2 + y^2 = r^2. \tag{20}$$

Since p equals to $0.5x_{\max}$, and $0.5x_{\max}$ is the radius r, then equation (20) can be written as follows:

$$x^2 - 2rx + y^2 = 0. \tag{21}$$

For a half of circle in Figure 2 above, equation (21) can be written

$$y = \sqrt{2rx - x^2}. \tag{22}$$

Let we start analysing equations of motion on X-axis direction to obtain the equations of motion on Y-axis as a function of the time t. There are two possible assumptions we will use:

(1) If the equation of motion on the X-axis direction is ULM, then Newton's first law will be valid, $\sum F_x = 0$ for constant mass. But, the normal force that leads to the center of the circle causes a force on the X-axis direction, $N_x = N \cos\theta$ in which

$N_x \neq 0.$ To defend Newton's first law is still valid in this assumption, it would require inertia on the *X*-axis direction that opposite to N_x direction but the magnitude is the same with N_x, call it (M_x), so that $N_x - M_x = 0.$ But inertia M_x does not come from the centrifugal force like a case in horizontal circular motion. If M_x comes from centrifugal force, the upward direction on *Y*-axis will have projection of the centrifugal force, $N_y = N \sin \theta.$ It is equal to the normal force on the downward direction, thus the equations of motion on *Y*-axis become ULM. It is impossible on vertical circular motion on earth surface.

(2) If the equation of motion on *X*-axis direction is UALM, then Newton's second law will be valid, $\sum F_x = ma_x$ for constant mass. We do not need any additional inertia on the *X*-axis direction because there is an acceleration a_x such that $N_x = N \cos \theta = ma_x.$ The problem here, if we assume the equation of motion on *X*-axis direction is UALM, equation (22) will become very complicated. The second assumption is also not valid because the value of *N* is not constant for every point, so that a_x not constant too. Therefore, we will assume that equation on *X*-axis is ULM with additional inertia M_x.

Substituting $x = v_0 \cos \alpha$ into (22), we have:

$$y(t) = \sqrt{2rv_0 \cos \alpha\, t - (v_0 \cos \alpha\, t)^2}. \tag{23}$$

The boundary condition is $0 \leq t \leq \dfrac{2r}{v_0 \cos \alpha}$, velocity component on *Y*-axis is

$$\frac{dy(t)}{dt} = v_y = \frac{2rv_0 \cos \alpha - 2v_0{}^2 \cos^2 \alpha\, t}{2y} \tag{24}$$

and component of acceleration is:

$$\frac{d^2 y(t)}{dt^2} = a_y = \frac{-4v_0{}^2 \cos^2 \alpha y^2 - (2rv_0 \cos \alpha - 2v_0{}^2 \cos^2 \alpha\, t)^2}{4y^{\frac{3}{2}}}. \tag{25}$$

The boundary conditions for velocity and acceleration are $0 < t < \dfrac{2r}{v_0 \cos \alpha}$.

In mathematics, semicircular trajectory also can be expressed as follows:

$$s = \theta r \tag{26}$$

with s is the length of trajectory or the distance travelled on the periphery of the orbit.

$$\theta = \omega t, \text{ with } \omega = \frac{\pi}{t}, \tag{27}$$

$$\frac{d\theta}{dt} = \omega = \frac{\pi \text{ radian}}{\text{time for a half of round}} = \frac{v}{r}, \tag{28}$$

$$v = \omega r = \frac{\pi}{t} r, \tag{29}$$

$$a_s = v\frac{d\theta}{dt} = \frac{v^2}{r} = \omega^2 r. \tag{30}$$

Linear speed in (29) can be expressed as follows:

$$v = \sqrt{{v_x}^2 + {v_y}^2}. \tag{31}$$

From equation (31), if v is not constant, then tangential acceleration will not equal to zero and centripetal acceleration in equation (30) will change from point to point. It makes velocity components on X-axis (v_x) or Y-axis (v_y) changes too. On the other hand, if v is constant, then tangential acceleration will equal to zero. But, theoretically, from the results of semicircular motion in Figure 2, linear speed v will not be constant because the velocity component on Y-axis direction is not constant.

Now we will review the equations of motion in Figure 2 using Newton's first and second laws of motion. On Y-axis direction, we use Newton's second law as follows:

$$\sum F_y = ma_y. \tag{32}$$

The normal force (N) at every point always leads to the center, while the weight of the object is always downward on Y-axis direction. By taking any point A arbitrary in Figure 2, we express the net force on Y-axis using equation (32) as follows:

$$-N_A \cos(90 - \theta) - mg = ma_y, \tag{33}$$

$$-N_y - mg = ma_y, \tag{34}$$

$$a_y = -\frac{N_y + mg}{m}. \tag{35}$$

From equation (35), we can say that it is not a constant value due to the normal force *N* changes from point to point. The negative sign states that the direction of the forces are downward. This result corresponds to equation (25) which states that a_y also changes for every point.

Integrating with respect to time *t* on equation (35), we have:

$$v_y = \int a_y dt = -\int \frac{N_y + mg}{m} dt, \tag{36}$$

$$v_y = -\int a_y dt = -gt - \int \frac{N_A \cos(90-\theta)}{m} dt. \tag{37}$$

We have a radial force at point *A* as follows:

$$N_A + mg\cos(90-\theta) = \frac{mv^2}{r}, \tag{38}$$

$$N_A = \frac{mv^2}{r} - mg\cos(90-\theta). \tag{39}$$

Substituting (39) into (37), we have:

$$v_y = C - \frac{v^2\cos(90-\theta)t}{r} - gt(1-\cos^2(90-\theta)), \tag{40}$$

$$v_y = C - \frac{v^2\cos(90-\theta)t}{r} - gt(\sin^2(90-\theta)). \tag{41}$$

By entering initial condition $t = 0,$ we have:

$$v_y = v_0 \sin\alpha - \left(\frac{v^2\cos(90-\theta)}{r} + g(\sin^2(90-\theta)\right)t. \tag{42}$$

Equation for position can be written as follows:

$$y = \int v_y dt.$$

By entering initial condition $t = 0,$ we have

$$y = v_0 \sin\alpha t - 0.5\left(\frac{v^2\cos(90-\theta)}{r} + g(\sin^2(90-\theta)\right)t^2. \tag{43}$$

We have assumed that the equation of motion on the *X*-axis direction is ULM, so, substituting $x = v_0 \cos \alpha t$ into (43)

$$y = (\tan \alpha)x - 0.5\left(\frac{v^2 \cos(90 - \theta)}{v_0^2 \cos^2 \alpha r} + \frac{g(\sin^2(90 - \theta))}{v_0^2 \cos^2 \alpha}\right)x^2. \quad (44)$$

3. Results and Discussion

Parabolic	**Semicircular**
On X-axis (ULM): $x = v_0 \cos \alpha t$ $v_x = v_0 \cos \alpha$ On *Y*-axis (UALM): $y = v_0 \sin \alpha t - 0.5\text{gt}^2$ $v_y = v_0 \sin \alpha - \text{gt}$ $a_y = -g$ Trajectory equation on two-dimensional axis : $y = (\tan \alpha)x - \left(\frac{g}{2v_0^2\alpha}\right)x^2$	On X-axis (ULM): $x = v_0 \cos \alpha t$ $v_x = v_0 \cos \alpha$ On *Y*-axis $y = v_0 \sin \alpha t - 0.5\left(\frac{v^2 \cos(90 - \theta)}{r} + (\sin^2(90 - \theta)\right)t^2$ With boundary condition: $r \neq 0$ $v_y = v_0 \sin \alpha - \left(\frac{v^2 \cos(90 - \theta)}{r} + g(\sin^2(90 - \theta)\right)t$ $a_y = -\frac{N_y + mg}{m}$ With boundary condition for velocity and acceleration $r \neq 0$ Trajectory equation on two-dimensional axis $y = (\tan \alpha)x - 0.5\left(\frac{v^2 \cos(90 - \theta)}{v_0^2 \cos^2 \alpha r} + \frac{g(\sin^2(90 - \theta))}{v_0^2 \cos^2 \alpha}\right)x^2$

Based on the results above, a semicircular motion is combinations of ULM on *X*-axis direction and a motion with changing acceleration on *Y*-axis. While the parabolic motion is combinations of ULM on the *X*-axis direction and UALM on the *Y*-axis.

Semicircular motion can be viewed as a special case of parabolic motion where the coefficient of highest order changes from time to time. While the parabolic motion, the coefficient of highest order is constant.

The unique case on the table above, Trajectory equation on two-dimensional axis for semicircular motion is similar with parabolic, but it is not differentiable if radius is zero. While in mathematics parabolic must be differentiable for every point arbitrary.

4. Conclusion

Several important points can be inferred from the results above:

1. Semicircular motion is a combination of ULM on the direction of the *X*-axis and a motion with changing acceleration on the *Y*-axis, while the parabolic motion is a combination of ULM on the direction of the *X*-axis and UALM on the *Y*-axis.

2. Semicircular motion can be viewed as a special case of parabolic motion where the coefficient of highest order changes from time to time, while the coefficient of highest order in parabolic motion is constant.

3. If an object is thrown upward obliquely near the earth's surface and trajectory we want is a semicircular, it requires:

a. on *X*-axis, the equations should be ULM,

b. on *Y*-axis, the equations should be a motion with changing acceleration,

c. there is additional inertia M_x against normal force N_x.

4. The equation of motion on the *Y*-axis using mathematical concepts and Newton's laws can be viewed as a regression non linear in statistics with the general form as follows:

$$Y = \hat{Y} + e, \; e \sim N(0, V),$$

where $Y =$ Ideal models represented by a circle and a parabolic equations, $\hat{Y} =$ Estimation model using Newton's laws of motion, $e =$ annoying factors or some other forces that we did not consider in this model.

For a parabolic motion (on earth's surface):

Position:

$$y = \frac{-\tan\theta}{x_s} v_0{}^2 \cos^2 \alpha t^2 + (2\tan\theta) v_0 \cos \alpha t,$$

$$\hat{y} = v_0 \sin \alpha t - 0.5 g t^2.$$

Velocity:

$$v_y = (2\tan\theta)v_0\cos\alpha - \frac{2\tan\theta}{x_s}v_0^2\cos^2\alpha t,$$

$$\hat{v}_y = v_0\sin\alpha - gt.$$

Acceleration:

$$a_y = -\frac{2\tan\theta}{x_s}v_0^2\cos^2\alpha,$$

$$\hat{a}_y = -g.$$

For a semicircular motion (on earth's surface):

Position:

$$y = \sqrt{2rv_0\cos\alpha t - (v_0\cos\alpha t)^2},$$

$$\hat{y} = v_0\sin\alpha t - 0.5\left(\frac{v^2\cos(90-\theta)}{r} + g(\sin^2(90-\theta)\right)t^2.$$

Velocity:

$$v_y = \frac{2rv_0\cos\alpha - 2v_0^2\cos^2\alpha t}{2y},$$

$$\hat{v}_y = v_0\sin\alpha\left(\frac{v^2\cos(90-\theta)}{r} + g(\sin^2(90-\theta)\right)t.$$

Acceleration:

$$a_y = \frac{-4v_0^2\cos^2\alpha y^2 - (2rv_0\cos\alpha - 2v_0^2\cos^2\alpha t)^2}{4y^{\frac{3}{2}}},$$

$$\hat{a}_y = -\frac{N_y + mg}{m}.$$

5. If we throw an object upward obliquely near the earth's surface, the most possible of trajectory is parabolic.

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