Fundamental Journal of Mathematics and Mathematical Sciences
Vol. 7, Issue 1, 2017, Pages 25-31
This paper is available online at http://www.frdint.com/
Published online January 11, 2017

DING PROJECTIVE DIMENSION

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Abstract

In basic homological algebra, the projective dimensions of modules play an important and fundamental role. In this paper, the closely related Ding projective dimensions are studied.

1. Introduction

Throughout the paper, R is a commutative ring with identity element, and all R-module are unital. We use $\mathcal{P}(\mathcal{R})$ and $\mathcal{F}(\mathcal{R})$ to denote the class of all projective and

Keywords and phrases: Ding projective dimension, Ding projective modules.

2010 Mathematics Subject Classification: 16D40, 18G20.

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Received December 18, 2016; Accepted January 4, 2017

flat R-module, respectively. For any module M, we use $pd_R(M)$ to denote projective dimensions of M.

In [5], the author introduced strongly Gorenstein flat module and strongly Gorenstein flat dimension, which are defined as follows:

Definition 1.1 [5]**.** Let n be a positive integer. An R-module M is called strongly Gorenstein flat module (we called Ding projective module) if there is an exact sequence

$$\mathbb{P} \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective right R-modules with $M = ker(P^0 \to P^1)$ such that $Hom(-, \mathfrak{flat})$ leaves the sequence exact.

Definition 1.2 [5]**.** For a right R-module M, let $SGfd(M)$ (we called $Dpd(M)$) denote the infimum of the set of n such that there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ of right R-modules, where each G_i is a strongly Gorenstein flat and called $SGfd(M)$, the strongly Gorenstein flat dimension of M (we called Ding projective dimension).

The main purpose of this paper is to study some properties of Ding projective dimension and we get some interesting results.

2. Ding Projective Dimension

Proposition 2.1. *Let M be an R-module with finite Ding projective dimension n. Then M admits a surjective Ding projective precover $\varphi : G \to M$, where $K = Ker(\varphi)$ satisfies $pd_R(K) = n - 1$.*

Proof. Pick an exact sequence, $0 \to K' \to P_{n-1} \to \cdots \to P_0 \to M \to 0$, where $P_0, \ldots, P_{n-1}$ are projective modules. Then K' is Ding projective by [11, Corollary 1.26]. Hence there is an exact sequence $0 \to K' \to Q^0 \to \cdots \to Q^{n-1} \to G' \to 0$, where $Q^0, \ldots, Q^{n-1}$ are projective modules, G' is Ding projective, and such that the functor $Hom(-, Q)$ leaves this sequence exact, whenever Q is flat.

Thus there exist homomorphisms, $Q^i \to P_{n-1-i}$, for $i = 0, 1, ..., n-1$, and $G \to M$, such that the following diagram is commutative.

$$\begin{array}{ccccccccccccc}
0 & \to & K' & \to & Q^0 & \to & \cdots & \to & Q^{n-1} & \to & G' & \to & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K' & \to & P_{n-1} & \to & \cdots & \to & P_0 & \to & M & \to & 0
\end{array}$$

This diagram gives a chain map between complexes

$$\begin{array}{ccccccccccc}
0 & \to & Q^0 & \to & \cdots & \to & Q^{n-1} & \to & G' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_{n-1} & \to & \cdots & \to & P_0 & \to & M & \to & 0
\end{array}$$

which induces an isomorphism in homology. Its mapping cone is exact, and all the modules in it, except for $P_0 \oplus G'$ (which is Ding projective) are projective. Hence the kernel K of $\varphi : P_0 \oplus G' \to M$ satisfies $pd_R(K) \leq n-1$ and then necessarily $pd_R(K) = n-1$.

Since K has finite projective dimension, we have $Ext_R^1(Q', K) = 0$ for any Ding projective modules Q', by [11, Proposition 1.11] and thus the homomorphism

$$Hom_R(Q', \varphi) : Hom_R(Q', G) \to Hom_R(Q', M)$$

is surjective. Hence $\varphi : G \twoheadrightarrow M$ is the desired precover of M.

Proposition 2.2. *Let* $0 \to G' \to G \to M \to 0$ *be a short exact sequence where* G' *and G are Ding projective modules, and where* $Ext_R^1(M, Q) = 0$ *for all projective module Q. Then M is Ding projective.*

Proof. Since $Dpd_R(M) \leq n$, Proposition 2.1 above gives the existence of an exact sequence $0 \to Q \to \tilde{G} \to M \to 0$, where Q is projective, and $\tilde{G}$ is Ding projective. By our assumption $Ext_R^1(M, Q) = 0$, this sequence splits, and hence M is Ding projective by [11, Theorem 1.15]

According to the definition of Ding projective dimension and Ding projective module, with standard arguments, we immediately obtain the next two results:

Lemma 2.3. *Let* $0 \to K \to G \to M \to 0$ *be an exact sequence of R-module where G is Ding projective. If M is Ding projective, then so is K. Otherwise we get* $Dpd_R(K) = Dpd_R(M) - 1 \geq 0$.

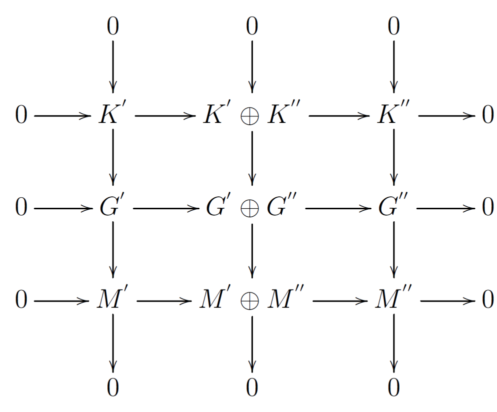
Lemma 2.4. *Consider an exact sequence* $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, *where* $G_0, \ldots, G_{n-1}$ *are Ding projective modules. Then* $Ext_R^i(K_n, L) \cong Ext_R^{i+n}(M, L)$ *for all R-modules L with finite flat dimension and all integers* $i > 0$.

Proposition 2.5. *If* $(M_\lambda)_{\lambda \in \Lambda}$ *is any family of R-modules, then we have an equality*

$$Dpd_R\left(\coprod M_\lambda\right) = \sup\{Dpd_R(M_\lambda) \mid \lambda \in \Lambda\}.$$

Proof. The inequality $\leq$ is clear, since $\mathcal{DP}(\mathcal{R})$ is closed under direct sums by [11, Theorem 1.15]. For the converse inequality $\geq$, it suffices to show that if M is any direct summand of an R-module M, then $Dpd_R(M') \leq Dpd_R(M)$. Naturally we may assume that $Dpd_R(M) = n$ is finite, and then proceed by induction on n.

The induction start is clear, because if M is Ding projective, then so is M' by [11, Theorem 1.15]. If $n > 0$, we write $M = M' \oplus M''$ for some module M''. Pick exact sequence $0 \to K' \to G' \to M' \to 0$ and $0 \to K'' \to G'' \to M'' \to 0$, where G' and G'' are Ding projective. We get a commutative diagram with split-exact rows



Applying Lemma 2.3 to the middle column in this diagram, we get that $Dpd_R(K' \oplus K'') = n-1$. Hence the induction hypothesis yields that $Dpd_R(K') \leq n-1$, and thus the short exact sequence $0 \to K' \to G' \to M' \to 0$ shows that $Dpd_R(M') \leq n$, as desired.

Theorem 2.6. *Let M be an R-module with finite Ding projective dimension, and let n be an integer. Then the following conditions are equivalent:*

(1) $Dpd_R(M) \leq n$;

(2) $Ext_R^i(M, L) = 0$ *for all* $i > n$, *and all R-modules L with finite* $fd_R(L)$;

(3) $Ext_R^i(M, Q) = 0$ *for all* $i > n$, *and all flat R-modules Q*;

(4) *For every exact sequence* $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, *where* $G_0, ..., G_{n-1}$ *are Ding projective modules, then also* K_n *is Ding projective module.*

Consequently, the Ding projective dimension of M is determined by the formulas:

$$Dpd_R M = \sup\{i \in \mathbb{N} \mid \exists L \in \overline{\mathcal{F}}(R) : Ext_R^i(M, L) \neq 0\}$$

$$= \sup\{i \in \mathbb{N} \mid \exists F \in \mathcal{F}(R) : Ext_R^i(M, F) \neq 0\}.$$

Proof. The proof is 'cyclic'. Obviously (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1), so we only have to prove the last two implications.

To prove (1) $\Rightarrow$ (2), we assume that $Dpd_R(M) \leq n$. By definition there is an exact sequence $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, where $G_0, ..., G_n$ are Ding projective modules. By Lemma 2.4 and [11, Proposition 1.11], we conclude the equalities $Ext_R^i(M, L) \cong Ext_R^{i-n}(G_n, L) = 0$ wherever $i > n$, and L has finite flat dimension, as desired.

To prove (3) $\Rightarrow$ (4), we consider an exact sequence

$$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

where $G_0, \ldots, G_{n-1}$ are Ding projective modules. Applying Lemma 2.4 to this sequence, and using the assumption, we get that $Ext_R^i(K_n, F) \cong Ext_R^{i+n}(M, F) = 0$ for every integer $i > n$, and every flat module F. Since $Dpd_R(M) < \infty$, by Lemma 2.3, we see that $Dpd_R(K_n) < \infty$. Hence there is an exact sequence $0 \to G'_m \to G'_{m-1} \to \cdots \to G'_0 \to K_n \to 0$, where $G'_0, \ldots, G'_m$ are Ding projective modules. We decompose it into short exact sequence, $0 \to C'_j \to G'_{j-1} \to C'_{j-1} \to 0$ for $j = 1, \ldots, m$, where $C'_m = G'_m$, $C'_0 = K_n$. Now another use of Lemma 2.4 gives that $Ext_R^1(C'_{j-1}, Q) \cong Ext_R^j(K_n, Q) = 0$ for all $j = 1, \ldots, m$, and all flat module Q. Thus Proposition 2.2 can be applied successively to conclude that $C'_0, \ldots, C'_m$ are Ding projective modules. In particular, $K_n = C'_0$ is Ding projective module.

The last formulas in the theorem for determination of $Dpd_R(M)$ are direct consequence of the equivalence between (1)-(3).

Proposition 2.7. *Let* $0 \to N' \to N \to N'' \to 0$ *be a short exact sequence of R-modules. If any two of the modules* N', N, N'' *have finite Ding projective dimension, then so has the third.*

Proof. By Theorem 2.6, the proof is obvious.

Proposition 2.8. *If M is an R-module with finite projective dimension, then* $Dpd_R(M) = pd_R(M)$.

Proof. Assume that $n = pd_R(M)$ is finite. By definition, there is always an inequality $Dpd_R(M) \leq pd_R(M)$, and consequently, we also have $Dpd_R(M) \leq n < \infty$. In order to show that $Dpd_R(M) = n$, we need, by Theorem 2.6, the existence of a flat module Q, such that $Ext_R^n(M, Q) \neq 0$. Since $pd_R(M) = n$, there is some module N, with $Ext_R^n(M, N) \neq 0$. Let Q be any flat module which is surjective onto N. From the long exact homology sequence, it now follows that also $Ext_R^n(M, Q) \neq 0$, as desired.

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