Fundamental Journal of Mathematics and Mathematical Sciences
Vol. 7, Issue 1, 2017, Pages 15-23
This paper is available online at http://www.frdint.com/
Published online January 10, 2017

CHATTERJEA CONTRACTION MAPPING THEOREM IN CONE HEPTAGONAL METRIC SPACE

CLEMENT BOATENG AMPADU

31 Carrolton Road
Boston, MA 02132-6303
USA
e-mail: drampadu@hotmail.com

Abstract

We introduce a concept of Cone Heptagonal Metric Space and obtain the Chatterjea Fixed Point Theorem (Chatterjea [1]) in this setting.

1. Introduction

Huang and Zhang [2] introduced the concept of a cone metric space. They replaced the set of real numbers by an ordered Banach space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have proved some fixed point theorems for different contractive type conditions in cone metric spaces; for examples, see, Common fixed point results for non commuting mappings without continuity in cone metric spaces (Abbas and Jungck [3]); Common fixed points for maps on cone metric space (Ilic and Rakocevic [4]); Some notes on the paper cone metric spaces and fixed point theorems of contractive mappings (Rezapour and Hamlbarani [5]).

Keywords and phrases: Chatterjea contraction, cone metric space, heptagonal property, fixed point theorems.

2010 Mathematics Subject Classification: 47H10, 54H25.

Received December 1, 2016; Accepted December 14, 2016

Garg [6] introduced the notion of cone hexagonal metric space and proved Banach contraction mapping principle in a normal cone hexagonal metric space setting. Very recently, Auwalu and Hincal [7] proved the Kannan contraction mapping principle in cone hexagonal metric space.

In this paper, inspired by the works of Garg [6] and Auwalu and Hincal [7], we introduce a concept of cone heptagonal metric space, and prove the Chatterjea contraction mapping principle in this setting.

This paper is organized as follows. Section 2 contains some preliminary ideas that would be useful in the sequel. Example 2.8 shows that the notion of cone heptagonal metric space is a proper extension of cone hexagonal metric space. Section 3 contains the main results, in particular, the Chatterjea contraction mapping principle in cone heptagonal metric space is given by Theorem 3.1, and Example 3.2 is given to illustrate the Chatterjea contraction principle in cone heptagonal metric space.

2. Preliminaries

Notation 2.1. E will denote a real Banach space.

Definition 2.2. $P \subset E$ will be called a cone iff

(a) P is closed, nonempty, and $P \neq \{0\}$,

(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ implies $ax + by \in P$,

(c) $x \in P$ and $-x \in P$ implies $x = 0$.

Notation 2.3. $\leq$ will denote a partial ordering with respect to P and will be defined by $x \leq y$ iff $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \operatorname{int}(P)$, where $\operatorname{int}(P)$ denotes the interior of P.

Definition 2.4. A cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$, the inequality $0 \leq x \leq y$ implies that $\|x\| \leq k\|y\|$. The least positive number k satisfying $\|x\| \leq k\|y\|$ is called the normal constant of P.

Remark 2.5. In this paper, we always assume that E is a real Banach space and P is a cone in E with $\operatorname{int}(P) \neq \Phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.6. Let X be a nonempty set. Suppose the mapping $d : X \times X \mapsto E$ satisfies

(a) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$,

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Remark 2.7. If we replace (c) of the previous definition with the following, which we call the heptagonal property, $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, v) + d(v, t) + d(t, y)$ for all $x, y, z, w, u, v, t \in X$ and for all distinct points $z, w, u, v, t \in X - \{x, y\}$, then we say d is a cone heptagonal metric on X, and we call (X, d) a cone heptagonal metric space.

Example 2.8. Let $X = \{r, s, t, u, v, w, k\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ be a cone in E. Define $d : X \times X \mapsto E$ by

$d(x, x) = 0$ for all $x \in X$,

$d(r, s) = d(s, r) = (6, 12)$,

$d(r, t) = d(r, u) = d(r, v) = d(r, w) = d(s, t) = d(s, u) = d(s, v) = d(s, w) = d(t, u) = d(t, v) = d(t, w) = d(u, v) = d(u, w) = d(v, w) = d(t, r) = d(u, r) = d(v, r) = d(w, r) = d(t, s) = d(u, s) = d(v, s) = d(w, s) = d(u, t) = d(v, t) = d(w, t) = d(v, u) = d(w, u) = d(w, v) = (1, 2)$,

$d(k, r) = d(k, s) = d(k, t) = d(k, u) = d(k, v) = d(k, w) = d(r, k) = d(s, k) = d(t, k) = d(u, k) = d(v, k) = d(w, k) = (5, 10)$.

Then it is easy to see that (X, d) is a cone heptagonal metric space, but it is not a cone hexagonal metric space, since it lacks the hexagonal property of Auwalu and Hincal [7], since $(6, 12) = d(r, s) > d(r, t) + d(t, u) + d(u, v) + d(v, w) + d(w, s) = (1, 2) + (1, 2) + (1, 2) + (1, 2) + (1, 2) = (5, 10)$ as $(6, 12) - (5, 10) = (1, 2) \in P$.

Definition 2.9. Let (X, d) be a cone heptagonal metric space. Let $\{x_n\}$ be a sequence in *X* and $x \in X$. If for every $c \in E$ with $0 << c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x\} << c$, then $\{x_n\}$ is said to be convergent.

Definition 2.10. Let (X, d) be a cone heptagonal metric space. Let $\{x_n\}$ be a sequence in *X*. If for every $c \in E$ with $0 << c$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) << c$, then $\{x_n\}$ is called a Cauchy sequence in *X*.

Definition 2.11. Let (X, d) be a cone heptagonal metric space. If every Cauchy sequence is convergent in *X*, then *X* will be called a complete cone heptagonal metric space.

Taking inspiration from Garg and Agarwal [8], we have the following

Lemma 2.12. *Let* (X, d) *be a cone heptagonal metric space, and P be a normal cone with normal constant k. Let* $\{x_n\}$ *be a sequence in X, then* $\{x_n\}$ *converges to x iff* $\|d(x_n, x)\| \to 0$ *as* $n \to \infty$.

Taking inspiration from Garg and Agarwal [8], we have the following

Lemma 2.13. *Let* (X, d) *be a cone heptagonal metric space*, and *P be a normal cone with normal constant k. Let* $\{x_n\}$ *be a sequence in X*, then $\{x_n\}$ *is a Cauchy sequence iff* $\|d(x_n, x_{n+m})\| \to 0$ as $n, m \to \infty$.

Taking inspiration from Jleli and Samet [9, Lemma 1.10], we have the following

Lemma 2.14. *Let* (X, d) *be a complete cone heptagonal metric space, P be a normal cone with normal constant k. Let* $\{x_n\}$ *be a Cauchy sequence in X and suppose there is a natural number N such that*

(a) $x_n \neq x_m$ *for all* $n, m > N$,

(b) x_n, x *are distinct points in X for all* $n > N$,

(c) x_n, y *are distinct points in X for all* $n > N$,

(d) $x_n \to x$ *and* $y_n \to y$ *as* $n \to \infty$.

Then $x = y$.

3. Main Results

Theorem 3.1. *Let* (X, d) *be a complete cone heptagonal metric space*, *P be a normal cone with normal constant k. Suppose the mapping* $f : X \mapsto X$ *satisfies the contractive condition*: $d(fx, fy) \leq \alpha[d(x, fy) + d(y, fx)]$ *for all* $x, y \in X$ *and* $\alpha \in \left(0, \frac{1}{2}\right)$. *Then*

(a) *f has a unique fixed point in X*,

(b) *for any* $x \in X$, *the iterative sequence* $\{f^n x\}$ *converges to the fixed point*.

Proof. Let $x \in X$. From the contractive condition, we deduce that

$$d(fx, f^2x) \leq \alpha d(x, f^2x) \leq \alpha[d(x, fx) + d(fx, f^2x)]$$

from which it follows that $d(fx, f^2x) \leq \frac{\alpha}{1-\alpha} d(x, fx)$. Similarly, we have, $d(f^2x, f^3x) \leq \frac{\alpha}{1-\alpha} d(fx, f^2x) \leq \left(\frac{\alpha}{1-\alpha}\right)^2 d(x, fx)$. Continuing, we deduce, for each positive integer n that, $d(f^n x, f^{n+1}x) \leq \left(\frac{\alpha}{1-\alpha}\right)^n d(x, fx) = r^n d(x, fx)$, where $0 \leq r := \frac{\alpha}{1-\alpha} < 1$. Now we divide the proof into two cases.

First Case. Let $f^m x = f^n x$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Let $m > n$.

Then $f^{m-n}(f^n x) = f^n x$, that is, $f^p y = y$, where $p = m - n$ and $y = f^n x$. Now since $p > 1$, we have $d(y, fy) = d(f^p y, f^{p+1} y) \leq r^p d(y, fy)$. Since $r \in [0, 1)$, it follows that $-d(y, fy) \in P$ and $d(y, fy) \in P$ which implies that $\|d(y, fy)\| = 0$, that is, $fy = y$.

Second Case. Assume that $f^m x \neq f^n x$ for all $m, n \in \mathbb{N}$ with $m \neq n$. Since $0 \leq r$, then it follows that $r^n \leq \frac{r^n}{1-r}$, thus it is clear that $d(f^n x, f^{n+1} x) \leq r^n d(x, fx) \leq \frac{r^n}{1-r} d(x, fx)$.

Now $d(f^n x, f^{n+2} x) \leq d(f^n x, f^{n+1} x) + d(f^{n+1} x, f^{n+2} x) \leq \frac{r^n}{1-r} d(x, fx) + \frac{r^{n+1}}{1-r} d(x, fx) = \frac{r^n(1+r)}{1-r} d(x, fx)$, and since $\frac{r^n}{1-r} d(x, fx) \leq \frac{r^n(1+r)}{1-r} d(x, fx)$, we deduce that $d(f^n x, f^{n+2} x) \leq \frac{r^n}{1-r} d(x, fx)$.

Also we have the following $d(f^n x, f^{n+3} x) \leq \frac{r^n(1 + r + r^2)}{1-r} d(x, fx)$, and since $\frac{r^n}{1-r} d(x, fx) \leq \frac{r^n(1 + r + r^2)}{1-r} d(x, fx)$, we deduce that $d(f^n x, f^{n+3} x) \leq \frac{r^n}{1-r} d(x, fx)$.

Therefore, it is clear, that we also have the following

$$d(f^n x, f^{n+4} x) \leq \frac{r^n}{1-r} d(x, fx) \quad \text{and} \quad d(f^n x, f^{n+5} x) \leq \frac{r^n}{1-r} d(x, fx).$$

Now if $m > 5$ is odd, then writing $m = 5 + 2l$, $l \geq 1$, and using the fact that $f^p x \neq f^r x$ for $p, r \in \mathbb{N}$, $p \neq r$, we see that

$$d(f^n x, f^{n+m} x) \leq \frac{r^n(1 + r + r^2 + \cdots + r^{2l})}{1-r} d(x, fx),$$

and since

$$\frac{r^n}{1-r} d(x, fx) \le \frac{r^n(1 + r + r^2 + \cdots + r^{2l})}{1-r} d(x, fx),$$

we deduce that $d(f^n x, f^{n+m} x) \le \frac{r^n}{1-r} d(x, fx)$.

Now if $m > 5$ is even, then writing $m = 2 + 2l$, $l \ge 2$, and using the fact that $f^p x \ne f^r x$ for $p, r \in \mathbb{N}$, $p \ne r$, we see that

$$d(f^n x, f^{n+m} x) \le \frac{r^n(1 + r + r^2 + \cdots + r^{2l-1})}{1-r} d(x, fx),$$

and since

$$\frac{r^n}{1-r} d(x, fx) \le \frac{r^n(1 + r + r^2 + \cdots + r^{2l-1})}{1-r} d(x, fx),$$

we deduce that $d(f^n x, f^{n+m} x) \le \frac{r^n}{1-r} d(x, fx)$.

Thus combining all the cases, we have $d(f^n x, f^{n+m} x) \le \frac{r^n}{1-r} d(x, fx)$, for all $n, m \in \mathbb{N}$.

Now if we take norm to inequality in the expression immediately above, we deduce that $\|d(f^n x, f^{n+m} x)\| \le k \frac{r^n}{1-r} \|d(x, fx)\|$. Since $k \frac{r^n}{1-r} \|d(x, fx)\| \to 0$ as $n \to \infty$, it follows that the sequence $\{f^n x\}$ is Cauchy, and by the completeness of X, there is $x^* \in X$ such that $f^n x \to x^*$ as $n \to \infty$. Now we show existence of the fixed point. Notice that

$$\begin{aligned} & d(x^*, fx^*) \\ & \le d(x^*, f^n x) + d(f^n x, f^{n+1} x) + d(f^{n+1} x, f^{n+2} x) + d(f^{n+2} x, f^{n+3} x) \\ & \quad + d(f^{n+3} x, f^{n+4} x) + d(f^{n+4} x, fx^*) \end{aligned}$$

$$\le d(x^*,\ f^n x) + d(f^n x,\ f^{n+1}x) + d(f^{n+1}x,\ f^{n+2}x) + d(f^{n+2}x,\ f^{n+3}x)$$

$$+ d(f^{n+3}x,\ f^{n+4}x) + \alpha[d(f^{n+3}x,\ fx^*) + d(x^*,\ f^{n+4}x)]$$

$$= d(x^*,\ f^n x) + d(f^n x,\ f^{n+1}x) + d(f^{n+1}x,\ f^{n+2}x) + d(f^{n+2}x,\ f^{n+3}x)$$

$$+ d(f^{n+3}x,\ f^{n+4}x) + \alpha d(x^*,\ f^{n+4}x) + \alpha d(f^{n+3}x,\ fx^*).$$

Taking limits in the above inequality, we get $d(x^*,\ fx^*) \le \alpha d(x^*,\ fx^*)$ and since $\alpha \in \left(0,\ \frac{1}{2}\right)$, it follows $\alpha < 1$, thus $1 - \alpha > 0$. Hence from the inequality $d(x^*,\ fx^*) \le \alpha d(x^*,\ fx^*)$, we deduce that $\|d(x^*,\ fx^*)\| = 0$, that is, $x^* = fx^*$. Now we show uniqueness of the fixed point. If y^* is another fixed point of *f*, then it follows that

$$d(x^*,\ y^*) = d(fx^*,\ fx^*) \le \alpha[d(x^*,\ fy^*) + d(y^*,\ fx^*)] = \alpha[2d(x^*,\ y^*)],$$

and since $\alpha \in \left(0,\ \frac{1}{2}\right)$, it follows that $1 - 2\alpha > 0$. Hence from the inequality $d(x^*, y^*) \le 2\alpha d(x^*,\ y^*)$, we deduce that $\|d(x^*, y^*)\| = 0$, that is, $x^* = y^*$, and uniqueness follows.

Now we illustrate the main result with the following

Example 3.2. Let $E = \mathbb{C}$ and $P = \{(x,\ y) := x + iy \mid x,\ y \in \mathbb{R},\ x,\ y \ge 0\}$ be a normal cone in *E*. Let *X* and $d : X \mapsto X$ be given by Example 2.8. Then as Example 2.8 shows, $(X,\ d)$ is a cone heptagonal metric space, but it is not a cone hexagonal metric space, since it lacks the hexagonal property of Auwalu and Hincal [7] since $6 + 12i = (6,\ 12) = d(r,\ s) > d(r,\ t) + d(t,\ u) + d(u,\ v) + d(v,\ w) + d(w,\ s)$ $= 5 + 10i = (5,\ 10)$ as $(6 + 12i) - (5 + 10i) = (6,\ 12) - (5,\ 10) = (1,\ 2) = 1 + 2i \in P$. Now define a mapping $f : X \mapsto X$ as follows, $fx = 6$, if $x \ne k$; $fx = 1$ if $x = k$. Note that *f* is not a contractive mapping with respect to the standard metric since $|fk - fs| = |1 - 6| = 5 = |k - s| = |7 - 2|$. However, *f* satisfies $d(fx,\ fy) \le \alpha[d(x,\ fy)$ $+d(y,\ fx)]$ for all $x,\ y \in X = \{r,\ s,\ t,\ u,\ v,\ w,\ k\} := \{1, 2, 3, 4, 5, 6, 7\}$ with

$\alpha = \frac{1}{6}$. Applying the previous theorem, we obtain that f admits the unique fixed point $x^* = w = 6$.

References

[1] S. K. Chatterjea, Fixed point theorems, C. R. Acad. Bulgare Sci. 25 (1972), 727-730.

[2] L. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332(2) (2007), 1468-1476.

[3] M. Abbas and G. Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341(1) (2008), 416-420.

[4] D. Ilic and V. Rakocevic, Common fixed points for maps on cone metric space, J. Math. Anal. Appl. 341(2) (2008), 876-882.

[5] S. Rezapour and R. Hamlbarani, Some notes on the paper cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 345(2) (2008), 719-724.

[6] M. Garg, Banach contraction principle on cone hexagonal metric space, Ultra Scientist 26(1) (2014), 97-103.

[7] A. Auwalu and E. Hincal, The Kannan's fixed point theorem in a cone hexagonal metric spaces, Adv. Res. 7(1) (2016), 1-9.

[8] M. Garg and S. Agarwal, Banach contraction principle on cone pentagonal metric space, J. Adv. Stud. Topol. 3 (2012), 12-18.

[9] M. Jleli and B. Samet, The Kannan's fixed point theorem in a cone rectangular metric space, J. Nonlinear Sci. Appl. 2 (2009), 161-167.