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ANALYSIS OF AN ECO-EPIDEMIOLOGICAL MODEL WITH DISEASE IN THE PREY AND PREDATOR

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Abstract

We analyze and formulate an Eco-Epidemiological model with disease in the preys and predators. We study the existence of the non-negative equilibria, obtain sufficient conditions of local asymptotical stability of the equilibria by the Hurwitz criterion, then we analyze the global stability of the positive equilibria by constructing appropriate Lyapunov functions.

1. Introduction

Mathematical ecology and mathematical epidemiology are major fields of study. Since transmissible disease in ecological situation can not be ignored, it is very important from both the ecological and the mathematical points of view to study ecological systems subject to epidemiological factors. A large number of studies have been performed in this field. However, all the papers available only discussed the disease spread in a species, see [1, 2] deal with the disease spreading among the predator population only, but in literatures [3-6] disease spreading among the preys

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population is considered. In our common life, the disease may spread among the prey and the predator. On the basic of this, the present paper deals with the prey-predator model with diseases in the preys and predators, and we suppose the predators with disease do not capture on the preys, the susceptible predators capture both on the susceptible and on the infected preys, but the capture rate is different, which is much closer to the actual situation. This paper considers the model as follows:

$$\begin{cases} \dfrac{dS_x}{dt} = rS_x\left[1-\left(\dfrac{S_x+I_x}{K}\right)\right] - \beta_1 S_x I_x - k_1 S_x S_y, \\ \dfrac{dI_x}{dt} = \beta_1 S_x I_x - k_2 I_x S_y - d_0 I_x, \\ \dfrac{dS_y}{dt} = k_1 \theta S_x S_y - d_1 S_y - \beta_2 S_y I_y + k_2 \theta I_x S_y, \\ \dfrac{dI_y}{dt} = \beta_2 S_y I_y - d_2 I_y, \end{cases}$$

where S_x, I_x, S_y, I_y are the densities of the susceptible preys, infected preys, susceptible predators and infected predators, r stands for the intrinsic growth rate of the susceptible preys, β_1, β_2 represent the transmission rate of the susceptible preys and susceptible predators, respectively, K is the environmental carrying capacity of the prey population, k_1 and k_2 represent the capturing rate of susceptible predators on the susceptible preys and on the infected preys, respectively, θ is the conversion of the predators, d_0 and d_2 are the death of infected preys and infected predators because of diseases, d_1 be the natural mortality of the susceptible preys.

All the parameters are assumed to be positive.

2. Equilbria Analysis

Let

$$\begin{cases} P(S_x, I_x, S_y, I_y) = rS_x\left[1-\left(\dfrac{S_x+I_x}{K}\right)\right] - \beta_1 S_x I_x - k_1 S_x S_y = 0, \\ Q(S_x, I_x, S_y, I_y) = \beta S_x I_x - k_2 I_x S_y - d_0 I_x = 0, \\ R(S_x, I_x, S_y, I_y) = k_1 \theta S_x S_y - d_1 S_y - \beta_2 S_y I_y + k_2 \theta I_x S_y = 0, \\ M(S_x, I_x, S_y, I_y) = \beta 2 S_y I_y - d_2 I_y = 0. \end{cases}$$

Case I. It is obvious that the system has non-negative equilibria points

$E_0(0, 0, 0, 0)$, and $E_1(K, 0, 0, 0)$.

Case II. $S_y = 0$, by

$$\begin{cases} S_x r\left(1 - \frac{S_x + I_x}{K}\right) - \beta_1 S_x I_x = 0, \\ I_x(\beta_1 S_x - d_0) = 0, \\ - d_2 I_y = 0, \end{cases}$$

we obtain: $I_y = 0$, $S_x = \frac{d_0}{\beta_1} = S_{x2}$, $I_x = \frac{r(\beta_1 K - d_0)}{\beta_1(\beta_1 K + r)} = I_{x2}$. $E_2(S_x, I_x, 0, 0)$ is non-negative equilibria point.

Case III. $I_y = 0$, we obtain

we obtain

$$S_y = \frac{[\beta_1 \theta K(\beta_1 d_0 + k_2 r) + \beta_1 r d_1] - d_0 \theta(K\beta_1^2 + 2k_2 r)}{k_2 \theta(K\beta_1^2 + 2k_2 r)},$$

$$S_x = \frac{\theta K(\beta_1 d_0 + k_2 r)}{K\theta\beta_1^2 + 2k_2 \theta r} = S_{x3},$$

$$I_x = \frac{d_1(K\beta_1^2 + 2k_2 r - k_1 r) - \theta K k_1(\beta_1 d_0 + k_2 r)}{k_2(K\theta\beta_1^2 + 2k_2 \theta r)} = I_{x3}.$$

We obtain non-negative equilibria point $E_3(S_{x3}, I_{x3}, S_{y3}, 0)$ when

$$d_1(K\beta_1^2 + 2k_2 r - k_1 r) > \theta K k_1(\beta_1 d_0 + k_2 r),$$

$$\beta_1 \theta K(\beta_1 d_0 + k_2 r) + \beta_1 r d_1 > d_0 \theta(K\beta_1^2 + 2k_2 r).$$

Case IV. $S_x \neq 0$, $I_x \neq 0$, $S_y \neq 0$, $I_y \neq 0$, we get

$$S_x = \frac{kd_2 + \beta_2 d_0}{\beta_1 \beta_2} = S_x^*,$$

$$I_x = \frac{r(K\beta_1\beta_2 - d_0\beta_2 - k_2 d_2) - k_1 \beta_1 d_2}{(r + \beta_1)\beta_1 \beta_2} = I_x^*,$$

$$S_y = \frac{d_2}{\beta_2} = S_y^*,$$

$$I_y = \frac{k_1\theta(d_0\beta_2 + d_0\beta_1\beta_2 + k_2 d_2 r) + k_2\theta r(K\beta_1\beta_2 - d_0\beta_2 - k_2 d_2)}{\beta_1\beta_1^2(r + \beta_1)} = I_y^*.$$

We obtain non-negative equilibria point $E_4(S_x^*, I_x^*, S_y^*, I_y^*)$ when $r(K\beta_1\beta_2 - d_0\beta_2 - k_2 d_2) > k_1\beta_1 d_2$.

3. Stability Analysis

The Jacobi matrix of the system (1) is

$$J = \begin{vmatrix} r - \frac{2r}{K}S_x - \left(\frac{r}{K} + \beta_1\right)I_x - k_1 S_y & \left(-\frac{r}{K} - \beta_1\right)S_x & -k_1 S_x & 0 \\ \beta_1 I_x & \beta_1 S_x - k_2 S_y - d_0 & -k_2 I_x & 0 \\ k_1\theta S_y & k_2\theta S_y & k_1\theta S_x - d_1 - \beta_2 I_y + k_2\theta I_x & -\beta_2 S_y \\ 0 & 0 & \beta_2 I_y & \beta_2 S_y - d_2 \end{vmatrix}.$$

Case I. For $E_0(0, 0, 0, 0)$, the characteristic equation is

$$(\lambda - r)(\lambda + d_0)(\lambda + d_1)(\lambda + d_2) = 0.$$

We can get the characteristic root as follows:

$$\lambda_1 = r > 0, \quad \lambda_2 = -d_0 < 0, \quad \lambda_3 = -d_1 < 0, \quad \lambda_4 = -d_2 < 0.$$

Therefore $E_0(0, 0, 0, 0)$ is a Saddle point.

Case II. For $E_1(K, 0, 0, 0)$, the characteristic equation is

$$(\lambda - r)(\lambda - \beta_1 K + d_0)(\lambda + d_1)(\lambda + d_2) = 0.$$

Roots of this equation are

$$\lambda_1 = -r < 0, \quad \lambda_2 = \beta_1 K - d_0, \quad \lambda_3 = -d_1 < 0, \quad \lambda_4 = -d_2 < 0.$$

Here $\lambda_2 < 0$ when $\beta_1 K < d_0$. Hence $E_1(K, 0, 0, 0)$ is locally asymptotically stable.

Case III. For $E_2(S_{x2}, I_{x2}, 0, 0)$, the Jacobi matrix is

$$J_{E_2} = \begin{vmatrix} -\frac{r}{K}S_{x2} & \left(-\frac{r}{K}-\beta_1\right)S_{x2} & 0 & 0 \\ \beta_1 I_{x2} & 0 & -k_2 I_{x2} & 0 \\ 0 & 0 & k_1\theta S_{x2} + k_2\theta I_{x2} & 0 \\ 0 & 0 & 0 & -d_2 \end{vmatrix}.$$

The corresponding characteristic polynomial is

$$D_2(\lambda) = (\lambda + d_2)(\lambda - k_1\theta S_{x2} - k_2\theta I_{x2} + d_1)$$

$$\times\left[\lambda^2 + \frac{r}{K}S_{x2}\lambda + \left(\frac{r}{K}+\beta_1\right)\beta_1 S_{x2} I_{x2}\right].$$

Because $\frac{r}{K}S_{x2} > 0$, $\left(\frac{r}{K}+\beta_1\right)\beta_1 S_{x2} I_{x2} > 0$, so λ_3, λ_4 have negative real parts.

When $k_1\theta S_{x2} + k_2\theta I_{x2} < d_1$, $\lambda_2 < 0$. So $E_2(S_{x2}, I_{x2}, 0, 0)$ is locally asymptotically stable.

Case IV. For $E_3(S_{x3}, I_{x3}, S_{y3}, 0)$, the Jacobi matrix is

$$J_{E_3} = \begin{vmatrix} -\frac{r}{K}S_{x3} & \left(-\frac{r}{K}-\beta_1\right)S_{x3} & 0 & 0 \\ \beta_1 I_{x3} & 0 & -k_2 I_{x3} & 0 \\ 0 & k_2\theta S_{y3} & 0 & -\beta_2 S_{y3} \\ 0 & 0 & 0 & \beta_2 S_{y3} - d_2 \end{vmatrix},$$

$$D_3(\lambda) = (\beta_2 S_{x3} - d_2 - \lambda)$$

$$\times\left\{-\lambda^3 - \frac{r}{K}S_{x3}\lambda^2 - \left[K^2\theta I_{x_3} S_{y3} + \beta_1\left(\frac{r}{K}+\beta_1\right)S_{x3}I_{x3}\right]\lambda - \frac{k_2^2\theta r}{K}S_{x3}I_{x3}S_{y3}\right\}.$$

It is clear that characteristic root $\lambda_1 = \beta_2 S_{y3} - d_2$.

Hence, we have the following main theorem:

Theorem 1. *$E_0(0, 0, 0, 0)$ is a saddle point. When $\beta_1 K < d_0$, $E_1(K, 0, 0, 0)$ is locally asymptotically stable. When $k_1\theta S_{x2} + k_2\theta I_{x2} < d_1$, $E_2(S_{x2}, I_{x2}, 0, 0)$ is locally asymptotically stable. When $\beta_2 S_{y3} < d_2$, $E_3(S_{x3}, I_{x3}, S_{y3}, 0)$ is locally asymptotically stable.*

4. Global Stability

Theorem 2. *The positive equilibria point $E_4 = (S_{x4}, I_{x4}, S_{y4}, 0)$ of the system is globally asymptotically stable.*

Proof. Take proper Lyapunov function $V(S_x, I_x, S_y, I_y): R_+^4 \to R$.

$$V(t) = m_1\left(S_x - S_{x4} - S_{x4}\ln\frac{S_x}{S_{x4}}\right) + m_2\left(I_x - I_{x4} - I_{x4}\ln\frac{I_x}{I_{x4}}\right)$$

$$+ m_3\left(S_y - S_{y4} - S_{y4}\ln\frac{S_y}{S_{y4}}\right) + m_4\left(I_y - I_{y4} - I_{y4}\ln\frac{I_y}{I_{y4}}\right).$$

Let

$$m_2\beta_1 - m_1\left(\frac{r}{K} + \beta 1\right) = 0, \quad m_3 k_1\theta - m_1 k_1 = 0,$$

$$k_2\theta m_3 - m_2 k_2 = 0, \quad m_4\beta_2 - m_3\beta_2 = 0.$$

Namely, when we take $m_2 = \frac{1}{\beta}\left(\frac{r}{K} + \beta_1\right)m_1,\ m_3 = m_4\frac{1}{\theta}m_1,\ m_1 > 0,$ then

$$\dot{V}(t) = -\frac{r}{K}m_1(S_x - S_{x4})^2 \le 0.$$

So by the LaSalle invariant set theorems, we know that $E_4(S_{x4}, I_{x4}, S_{y4}, I_{y4})$ is globally asymptotically stable.

5. Conclusion

This paper mainly discusses the prey-predator model with disease in the preys and predators, we get the conditions of local asymptotical stability and the existence of the boundary balance. We prove the positive balance point

$E_4(S_{x4}, I_{x4}, S_{y4}, I_{y4})$ is globally asymptotically stable by constructing Lyapunov function.

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