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AN UNEXPECTED TOPOLOGICAL CENSOR

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Abstract

Morris-Thorne wormholes with a cosmological constant Λ have been studied extensively, even allowing Λ to be replaced by a space variable scalar. These wormholes cannot exist, however, if Λ is both space and time dependent. Such a Λ will therefore act as a topological censor. While not likely to have a bearing on the present, possible cosmological consequences of inflation cannot be discounted.

1. Introduction

Wormholes are handles or tunnels in the spacetime topology connecting two separate and distinct regions of spacetime. These regions may be part of our Universe or of different universes. The pioneer work of Morris and Thorne [1] has shown that macroscopic wormholes may be actual physical objects, provided that certain energy conditions are violated. Several wormhole studies have added the cosmological constant Λ [2-4].

When Einstein first introduced the cosmological constant into his field equations in 1917, he was still striving for consistency with Mach's principle. From the standpoint of cosmology, however, Λ served to create a kind of repulsive pressure to yield a stationary Universe. Eventually Zel'dovich identified Λ with the vacuum energy density due to quantum fluctuations [5].

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It has been proposed from time to time that the "constant" is actually a variable parameter. For example, in discussing a family of asymptotically flat globally regular solutions to the Einstein field equations, Dymnikova [6] notes that the source term corresponds to an r-dependent Λ. Assuming that Λ does indeed have the form $\Lambda = \Lambda(r)$, Rahaman et al. [7] obtained a class of wormhole solutions, while Ray et al. [8] studied various models that can be applied to the classical electron of the Lorentz type. Cosmic strings with $\Lambda = \Lambda(r)$ are discussed in [9]. In [10] the variable Λ is derived from a higher spatial dimension and manifests itself as an energy-density for the vacuum.

Another widely discussed possibility is a space- and time-dependent Λ, i.e., $\Lambda = \Lambda(r, t)$, suggested by recent observations of high redshift Type Ia supernovae [11-15]. For a detailed discussion with an extensive list of references, see Alcaniz [16]. For various Λ-decay scenarios from the original high value during inflation to the present, see [17] and references therein. [18] discusses the big bang, as well as the "big bounce", referring to variable-Λ models having a non-singular origin.

Using a natural extension of a metric proposed by Delgaty and Mann [2], it is shown in this paper that if Λ is both space and time dependent, so that $\Lambda = \Lambda(r, t)$, then a wormhole of the Morris-Thorne type will have a curvature singularity at the center. Possible cosmological implications are discussed at the end.

2. Background

Using units in which $c = G = 1$, our starting point is the Einstein-de Sitter metric

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

which is the unique solution of the vacuum Einstein field equations for a spherically symmetric spacetime with a positive cosmological constant. The line element reduces to the Schwarzschild line element if $\Lambda = 0$. The wormhole metric in [1],

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

provides a motivation for the following metric, proposed by Delgaty and Mann [2]:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3}$$

In [2], Λ is fixed, while the constant 1 is incorporated in the function $M(r)$. This metric describes a traversable wormole in $(3+1)$ dimensions with a cosmological constant Λ. If Λ is to have the form $\Lambda = \Lambda(r, t)$, then equation (3) becomes the only natural choice for the new metric.

In the metric, equation (3), $\Phi(r)$ is called the *redshift function*. If $\Lambda = 0$, then $M(r) = b(r)$. So $M(r)$ will be called the *shape function*; thus $M(r_0) = r_0$. (Recall that in equation (2), the sphere of radius $r = r_0$ is the *throat* of the wormhole). Qualitatively, $M(r)$ has the form shown in Figure 1. Observe that Λ is a positive function of both *r* and *t*.

According to [19], since the wormhole described by the metric in equation (3) is dynamic, there are actually two throats on opposite sides of the *center* $r = r_1$. This center is determined implicitly (for any fixed *t*) from the equation

$$1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3} = 0. \tag{4}$$

(Observe that the entire sphere $r = r_1$ lies in the same time slice). After rearranging terms, we get for any fixed time-slice

$$F(r) = M(r) + \frac{1}{3}r^3\Lambda(r, t) = r. \tag{5}$$

So for any fixed *t*, a solution to equation (5) is a fixed point $F(r_1) = r_1$. (See Figure 1). Since $M, \Lambda,$ and r are all positive, $M(r_1) < r_1$. So $r_1 > r_0$. Since the sphere $r = r_1$ is the center of the wormhole, $r = r_0$ is not in the manifold, while each throat is a sphere with time-dependent radius $r_2 > r_1$.

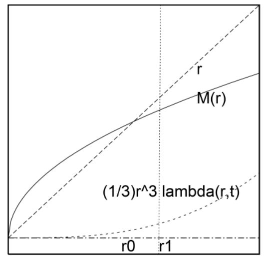

Figure 1. Graph showing the fixed point $r = r_1$ of $F(r)$.

3. The Failed Solution

To study the presumptive wormhole solution, it is necessary to compute the components of the Riemann curvature and Einstein tensors using the following orthonormal basis:

$$\theta^0 = e^{\phi(r)}dt, \quad \theta^1 = \left(1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3}\right)^{-1/2} dr,$$

$$\theta^2 = rd\theta, \quad \theta^3 = r\sin\theta d\phi.$$

Some of the components of the Einstein tensor are listed next:

$$G_{\hat{t}\hat{t}} = \frac{M'(r)}{r^2} + \Lambda(r, t) + \frac{1}{3} r \frac{\partial}{\partial r} \Lambda(r, t), \tag{6}$$

$$G_{\hat{r}\hat{r}} = \frac{2}{r}\left(1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3}\right)\Phi'(r) - \frac{M(r)}{r^3} - \frac{\Lambda(r, t)}{3}, \tag{7}$$

$$G_{\hat{r}\hat{t}} = \frac{r}{3} e^{-\Phi(r)} \frac{\partial}{\partial t} \Lambda(r, t)\left(1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3}\right)^{-1/2}. \tag{8}$$

From the Einstein field equations with cosmological constant,

$$G_{\hat{\alpha}\hat{\beta}} + \Lambda g_{\hat{\alpha}\hat{\beta}} = 8\pi T_{\hat{\alpha}\hat{\beta}}, \tag{9}$$

we obtain

$$T_{\hat{\alpha}\hat{\beta}} = \frac{1}{8\pi}(G_{\hat{\alpha}\hat{\beta}} + \Lambda g_{\hat{\alpha}\hat{\beta}}). \tag{10}$$

So

$$T_{\hat{t}\hat{t}} = \rho(r, t) = \frac{1}{8\pi}\left(\frac{M'(r)}{r^2} + \frac{1}{3} r \frac{\partial}{\partial r} \Lambda(r, t)\right), \tag{11}$$

$$T_{\hat{r}\hat{r}} = p(r, t) = \frac{1}{8\pi}\left[\frac{2}{r}\left(1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3}\right)\Phi'(r) - \frac{M(r)}{r^3} + \frac{2}{3}\Lambda(r, t)\right], \tag{12}$$

$$T_{\hat{r}\hat{t}} = T_{\hat{t}\hat{r}} = -f(r, t) = \frac{1}{8\pi}\frac{r}{3} e^{-\Phi(r)} \frac{\partial}{\partial t}\Lambda(r, t)$$

$$\times\left(1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3}\right)^{-1/2}, \tag{13}$$

where $f(r, t)$ is usually interpreted as the energy flux in the outward radial direction [20].

Now let us assume that at the throat $(r = r_2)$ the usual flare-out conditions have been met and that for every t the weak energy condition (WEC) has been violated. (The WEC states that given the stress-energy tensor $T_{\hat{\alpha}\hat{\beta}}$, the inequality $T_{\hat{\alpha}\hat{\beta}}\mu^{\hat{\alpha}}\mu^{\hat{\beta}} \geq 0$ holds for all time-like vectors and, by continuity, all null vectors). So for the radial outgoing null vector (1, 1, 0, 0) we therefore have

$$T_{\hat{\alpha}\hat{\beta}}\mu^{\hat{\alpha}}\mu^{\hat{\beta}} = \rho + p \pm 2f < 0. \tag{14}$$

In this manner all the conditions for the existence of a wormhole appear to have been met. However, the real problem does not depend on any violation of the WEC: in view of equation (4), we have for any given t

$$1 - \frac{M(r_1)}{r_1} - \frac{\Lambda(r_1, t)r_1^2}{3} = 0$$

at the center $r = r_1$. Hence $f(r, t)$ cannot be a finite quantity as long as $\partial\Lambda(r, t)/\partial t \neq 0$. Similarly, the components $G_{\hat{\theta}\hat{\theta}}$ and $G_{\hat{\phi}\hat{\phi}}$, which are proportional to the lateral pressure p_t, cannot be finite as long as Λ is time dependent:

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -e^{-2\Phi(r)}\left[\frac{r^2}{6}\frac{\partial^2}{\partial t^2}\Lambda(r, t)\left(1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3}\right)^{-1}\right.$$

$$+\frac{r^4}{12}\left(\frac{\partial}{\partial t}\Lambda(r,\,t)\right)^2\left(1-\frac{M(r)}{r}-\frac{\Lambda(r,\,t)r^2}{3}\right)^{-2}\Bigg]+\text{other terms.} \tag{15}$$

Finally, it is shown in [1] that for a wormhole to be traversable by humanoid travelers, the radial tidal constraint must be met: $\left|R_{\hat{t}\hat{r}\hat{r}}{}^{\hat{t}}\right| \leq (10^8\,\text{m})^{-2}$, where $R_{\hat{t}\hat{r}\hat{r}}{}^{\hat{t}}$ is a component of the Riemann curvature tensor. This component is given by

$$\begin{aligned}R_{\hat{t}\hat{r}\hat{r}}{}^{\hat{t}} = -e^{-2\Phi(r)}&\left[\frac{r^2}{6}\frac{\partial^2}{\partial t^2}\Lambda(r,\,t)\left(1-\frac{M(r)}{r}-\frac{\Lambda(r,\,t)r^2}{3}\right)^{-1}\right.\\&\left.+\frac{r^4}{12}\left(\frac{\partial}{\partial t}\Lambda(r,\,t)\right)^2\left(1-\frac{M(r)}{r}-\frac{\Lambda(r,\,t)r^2}{3}\right)^{-2}\right]\\&+\left(1-\frac{M(r)}{r}-\frac{\Lambda(r,\,t)r^2}{3}\right)(\Phi''(r)+[\Phi'(r)]^2)\\&-\frac{1}{2}\Phi'(r)\left(\frac{M'(r)}{r}-\frac{M(r)}{r^2}+\frac{2}{3}r\Lambda(r,\,t)+\frac{1}{3}r^2\frac{\partial}{\partial r}\Lambda(r,\,t)\right).\end{aligned} \tag{16}$$

Because of equation (4), we see that, once again, the right-hand side of equation (16) cannot be finite at the center as long as Λ is time dependent. The same problem arises with the lateral tidal constraints. So even if the earlier problems did not occur, the wormhole would not be traversable.

4. A Divergent Scalar Quantity

The singularities encountered so far could conceivably be removed by a suitable coordinate transformation, as, for example, in the Schwarzschild case. To show that the spacetime is singular, we need a scalar quantity that becomes infinite. To this end we list the components of the Ricci tensor. First we define the function

$$\begin{aligned}H(r) = &\left(1-\frac{M(r)}{r}-\frac{\Lambda(r,\,t)r^2}{3}\right)(-\Phi''(r)-[\Phi'(r)]^2)\\&+\frac{1}{2}\Phi'(r)\left(\frac{rM'(r)-M(r)}{r^2}+\frac{2}{3}r\Lambda(r,\,t)+\frac{1}{3}r^2\frac{\partial}{\partial r}\Lambda(r,\,r)\right).\end{aligned}$$

Then

$$R_{\hat{t}\hat{t}} = -H(r) + \frac{2}{r}\left(1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3}\right)\Phi'(r),$$

$$R_{\hat{r}\hat{r}} = H(r) + \frac{1}{r}\left(\frac{rM'(r) - M(r)}{r^2} + \frac{2}{3}r\Lambda(r, t) + \frac{1}{3}r^2\frac{\partial}{\partial r}\Lambda(r, t)\right),$$

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = -\frac{1}{r}\left(1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3}\right)\Phi'(r)$$
$$+ \frac{1}{2r}\left(\frac{rM'(r) - M(r)}{r^2} + \frac{2}{3}r\Lambda(r, t) + \frac{1}{3}r^2\frac{\partial}{\partial r}\Lambda(r, t)\right)$$
$$+ \frac{1}{r^2}\left(\frac{M(r)}{r} + \frac{\Lambda(r, t)r^2}{3}\right),$$

and

$$R_{\hat{r}\hat{t}} = \frac{r}{3}e^{-\Phi(r)}\frac{\partial}{\partial t}\Lambda(r, t)\left(1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3}\right)^{-1/2}.$$

Now consider the square of the curvature scalar

$$R_{\hat{\alpha}\hat{\beta}}R^{\hat{\alpha}\hat{\beta}} = R_{\hat{t}\hat{t}}R^{\hat{t}\hat{t}} + 2R_{\hat{r}\hat{t}}R^{\hat{r}\hat{t}} + R_{\hat{r}\hat{r}}R^{\hat{r}\hat{r}} + R_{\hat{\theta}\hat{\theta}}R^{\hat{\theta}\hat{\theta}} + R_{\hat{\phi}\hat{\phi}}R^{\hat{\phi}\hat{\phi}}.$$

For any fixed t (that is, for any fixed time-slice), the term $R_{\hat{r}\hat{t}}R^{\hat{r}\hat{t}}$ is divergent for some $r = r_1$ whenever

$$\frac{\partial}{\partial t}\Lambda(r, t) \neq 0.$$

Being a scalar quantity, it diverges at the center in all coordinate systems.

5. Discussion

Before discussing the various implications, let us first recall that the assumption $\frac{\partial}{\partial t}\Lambda(r, t) \neq 0$ has some clear-cut consequences: Equations (13), (15), and (16) imply that the energy flux, lateral pressure, and curvature cannot be finite at the center of the wormhole. Since the scalar quantity $R_{\hat{\alpha}\hat{\beta}}R^{\hat{\alpha}\hat{\beta}}$ also diverges, there is a curvature singularity at the center. So given the ansatz, equation (3), it follows that for a wormhole of the Morris-Thorne type to exist, Λ must not be time dependent. More formally, using the language of the topological censorship principle [21, 22], causal curves originating from and ending in a simply connected asymptotic region do not see any non-trivial topology and can therefore be deformed to a curve contained entirely within the asymptotic region. In the present situation, an ingoing radial null geodesic continues to move inward and so cannot pass through the wormhole and probe the topology. A time-dependent Λ will therefore act as a topological censor for wormholes of the Morris-Thorne type.

Returning to the line element (3), suppose $(\partial/\partial t)\Lambda(r, t) = 0$ for $t \leq t_0$ (for some t_0) and that a Morris-Thorne wormhole exists. If $(\partial/\partial t)\Lambda(r, t)$ becomes nonzero for $t > t_0$, then the center develops a curvature singularity. So the entire model, equation (3), breaks down and we no longer have a valid wormhole solution. With the properties of black holes in mind, this singularity is not likely to disappear even if Λ becomes constant again. Moreover, since all the points on the sphere are singularities, the infinite gravitational forces between them would pull the entire sphere into a single point, thereby producing a black hole.

While all these conclusions are based on fairly straightforward calculations, one can question their relevance: for if Λ really does change, then the rate of change is likely to be so minute as to be practically undetectable. Putting it another way, even if Λ should be independent of r, which is also likely, $(\partial/\partial t)\Lambda(r, t)$ is going to be zero within the margin of experimental error. So the outcome has no bearing on the present.

The situation would have been entirely different during a period when Λ really did change, at least with respect to time, as would have been the case during inflation. Here the existence of a kind of *vacuum energy* caused the Universe to act like an approximation to a de Sitter solution since it was dominated by a large effective cosmological "constant" ([23, p. 10]). At the very least, the change in Λ would have been very large at the beginning of inflation, as well as the end. Now, submicroscopic wormholes existing prior to the onset of inflation could conceivably have expanded to macroscopic size [20]. However, such wormholes could not have survived the beginning of inflation.

During inflation, Λ would not only have been large, but it may also have been constant. (If not, wormholes could not have formed). It is generally believed that inflation provides a possible explanation for the initial inhomogeneities that have led to the macroscopic structures we see today. These large-scale structures could have included wormholes. But since Λ changed again rapidly at the end of inflation, such wormholes could not have survived either.

These outcomes help explain why (apart from gravitational lensing) the stars and galaxies observed are not believed to be multiple images of a much smaller set: such a phenomenon would indeed require a multiply-connected spacetime. In addition, the possibility that previously existing wormholes had become black holes would help explain the large number of black holes discovered, while the evidence for the existence of wormholes is entirely lacking.

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