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ABOUT A SERIES OF UNSOLVED PROBLEMS IN THE THEORY OF STABILITY AND OPTIMAL CONTROL OF STOCHASTIC SYSTEMS

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Abstract

This paper continues a series of papers devoted to unsolved problems in the theory of stability and optimal control of stochastic systems. A delay differential equation with stochastic perturbations of the white noise and Poisson's jumps types is considered. In contrast to the known stability condition, in which it is assumed that stochastic perturbations fade on the infinity quickly enough, a new situation is studied, in which stochastic perturbations can fade on the infinity either slowly or not fade at all. Some unsolved problem in this connection is proposed to readers' attention.

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1. Introduction

The proposed here unsolved problem continues a series of unsolved problems in the theory of stability and optimal control for stochastic difference equations, stochastic differential equations and stochastic partial differential equations, that were presented during the recent years in some international conferences and papers (see [1-12]). All these problems still need to be solved.

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a complete probability space, $\{\mathfrak{F}_t\}_{t\geq 0}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e., $\mathfrak{F}_s \subset \mathfrak{F}_t$ for $s < t$, $\mathbf{E}$ be the expectation with respect to the measure $\mathbf{P}$, H_2 be the space of $\mathfrak{F}_0$-adapted stochastic processes $\varphi(s)$, $s \leq 0$, $\|\varphi\|^2 = \sup_{s\leq 0} \mathbf{E}|\varphi(s)|^2$.

Following Gikhman and Skorokhod [13], we will consider the stochastic delay differential equation

$$dx(t) = \left(Ax(t) + \sum_{i=1}^{k} B_i x(t - h_i)\right)dt + \sum_{i=1}^{m} C_i(t)x(t)dw_i(t)$$

$$+ \int G(t, u)x(t)\tilde{\nu}(dt, du), \quad t \geq 0, \tag{1}$$

$$x(s) = \phi(s) \in H_2, \quad s \in [-h, 0], \quad h = \max_{i=1,\ldots,k} h_i,$$

where $x(t) \in \mathbf{R}^n$, A, B, $C_i(t)$, $G(t, u)$ are $n \times n$-matrices, $h_i > 0$, $w_1(t), \ldots, w_m(t)$ are the standard Wiener processes, that are mutually independent and also independent of the Poisson measure $\nu(t, A)$,

$$\mathbf{E}\nu(t, A) = t\Pi(A), \quad \tilde{\nu}(t, A) = \nu(t, A) - t\Pi(A).$$

Definition 1.1 [14]**.** The zero solution of Equation (1) is called:

- mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$\mathbf{E}|\, x(t)\,|^2 < \varepsilon,\ t \geq 0,$ provided that $\|\, \phi\,\|^2 < \delta;$

- asymptotically mean square stable if it is mean square stable and $\lim_{t\to\infty} \mathbf{E}|\, x(t)\,|^2 = 0$ for each initial function $\phi(s)$.

Some particular cases of Equation (1) are considered in [15, 16], where it is proven that if the stochastic perturbations fade on the infinity quickly enough then the asymptotically stable zero solution of the corresponding deterministic system remains asymptotically mean square stable regardless of the level of these perturbations.

Below a situation is studied when stochastic perturbations can fade on the infinity either slowly or not fade at all. By that some unsolved problem is also proposed.

2. Equation without delays

Consider at first Equation (1) without delays, i.e., by the condition

$$B_i = 0, \quad i = 1, \ldots, k. \tag{2}$$

Let L be the generator of this equation [13, 14]. Then for the function $V(x(t)) = |\, x(t)\,|^2$ we have

$$\begin{aligned} LV(x(t)) &= 2x'(t)Ax(t) + \sum_{i=1}^{m} x'(t)C_i'(t)C_i(t)x(t) \\ &\quad + \int x'(t)G'(t,\, u)G(t,\, u)x(t)\Pi(du) \\ &= x'(t)[A + A' + Q(t)]x(t), \end{aligned} \tag{3}$$

where

$$Q(t) = \sum_{i=1}^{m} C_i'(t)C_i(t) + \int G'(t,\, u)G(t,\, u)\Pi(du).$$

Let $\rho(t) = \| Q(t) \|$ be the norm of the matrix $Q(t)$, i.e.,

$$x'Q(t)x \leq \rho(t)|x|^2. \tag{4}$$

Assume that $A + A'$ is a negative definite matrix, i.e.,

$$x'(A + A')x \leq -\alpha |x|^2, \quad \alpha > 0, \tag{5}$$

and, besides, suppose that

$$\sup_{t \geq 0} \rho(t) < \alpha \quad \text{or} \quad \int_0^\infty \rho(t)dt < \infty. \tag{6}$$

Put also

$$\mu(t) = \frac{1}{t}\int_0^t \rho(s)ds, \quad \mu = \limsup_{t \to \infty} \mu(t). \tag{7}$$

Remark 2.1. Note that if the first or the second condition (6) holds then, respectively, $\mu < \alpha$ or $\mu = 0 < \alpha$. But the inequality $\mu < \alpha$ can be hold even by the condition

$$\int_0^\infty \rho(t)dt = \infty. \tag{8}$$

For example, for the function $\rho(t) = \dfrac{2\alpha}{t+1}$ none from the conditions (6) are satisfied, but the both conditions $\mu = 0 < \alpha$ and (8) are obviously satisfied.

Theorem 2.1. *Let* α *and* μ, *defined in* (5) *and* (7), *satisfy the condition* $\mu < \alpha$. *Then the zero solution of Equation* (1) *with the condition* (2) *is asymptotically mean square stable.*

Proof. Using (3) and the definitions (4), (5) for $\rho(t)$ and α, we have

$$LV(x(t)) \leq (-\alpha + \rho(t))|x(t)|^2.$$

From this and Dynkin's formula [13]

$$\mathbf{E}V(x(t)) = \mathbf{E}V(x(0)) + \int_0^t \mathbf{E}LV(x(s))ds$$

for the function $V(x(t)) = |x(t)|^2$ it follows that

$$\frac{d}{dt}\mathbf{E}|x(t)|^2 = \mathbf{E}LV(x(t)) \le (-\alpha + \rho(t))\mathbf{E}|x(t)|^2$$

or

$$\frac{d\mathbf{E}|x(t)|^2}{\mathbf{E}|x(t)|^2} \le (-\alpha + \rho(t))dt.$$

Integrating this inequality and using (7), we obtain

$$\mathbf{E}|x(t)|^2 \le \mathbf{E}|x(0)|^2 \exp\left\{-\alpha t + \int_0^t \rho(s)ds\right\}$$

$$= \mathbf{E}|x(0)|^2 \exp\{(-\alpha + \mu(t))t\}.$$

From this and $\mu < \alpha$ it follows that $\mathbf{E}|x(t)|^2 \le c\mathbf{E}|x(0)|^2$ for some $c > 0$ and $\lim_{t\to\infty} \mathbf{E}|x(t)|^2 = 0$, i.e., the zero solution of Equation (1), (2) is asymptotically mean square stable. The proof is completed.

Remark 2.2. Note that the integrability of the function $\rho(t)$ is not a necessary condition for asymptotic mean square stability of the zero solution of the stochastic delay differential Equation (1). As it is shown in [16] for a simple scalar equation of the type of (1), (2)

$$dx(t) = -ax(t)dt + \sigma(t)x(t)dw(t) + \int \gamma(t, u)x(t)\tilde{\nu}(dt, du),$$

the zero solution can be asymptotically mean square stable even by the condition (8).

Unsolved problem. *The proof of asymptotic mean square stability of the zero solution of the stochastic delay differential Equation* (1) *under the condition* (8) *is currently an unsolved problem*, *which is offered to the attention of potential readers*.

3. Conclusions

To the readers attention an unsolved problem is offered about of stability of a delay differential equation under unfading or slowly fading stochastic perturbations. This unsolved problem complements a number of other previously published unsolved problems, the solutions to which have not until now been obtained.

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