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A NOTE ON THE PREDICTABILITY OF FLAT GALACTIC ROTATION CURVES

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Abstract

Based on an exact solution of the Einstein field equations, it is proposed in this note that the dark-matter hypothesis could have led to the prediction of flat galactic rotation curves long before the discovery thereof by assuming that on large scales the matter in the Universe, including dark matter, is a perfect fluid.

1. Introduction

It has been known since the 1930's that galaxies and clusters of galaxies do not have enough visible matter to account for observed motions. The missing matter was subsequently called "dark matter." The full implications of the existence thereof was not recognized until the 1970's, when it was observed that galaxies show solid-body rotations near the center but exhibit flat rotation curves sufficiently far from the galactic core. This indicates that the mass continues to increase linearly with the radius, attributable to the presence of dark matter. It is proposed in this note that flat

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rotation curves could have been predicted from Einstein's theory prior to 1970 by assuming that on large scales the matter in the Universe is a perfect fluid that also includes the hypothesized dark matter. So we take the equation of state to be $p = \omega\rho$ and then make use of the most general possible exact solution of the Einstein field equations.

2. Predictions from Exact Solutions

We start with the line element

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

using units in which $c = G = 1$. The Einstein field equations are

$$e^{-2\Lambda}\left(\frac{2\Lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi\rho, \tag{2}$$

$$e^{-2\Lambda}\left(\frac{1}{r^2} + \frac{2\Phi'}{r}\right) - \frac{1}{r^2} = 8\pi p, \tag{3}$$

and

$$e^{-2\Lambda}\left[(\Phi')^2 + \Phi'' - \Lambda'\Phi' + \frac{1}{r}(\Phi' - \Lambda')\right] = 8\pi p_t. \tag{4}$$

Since Eq. (4) can be obtained from the conservation of the stress energy tensor $T^{\mu\nu}{}_{;\nu} = 0$, only Eqs. (2) and (3) are needed.

Substituting in the equation of state $p = \omega\rho$, we obtain the following differential equation:

$$-\omega\Lambda' = -\Phi' + \frac{1}{2r}(e^{2\Lambda} - 1)(\omega + 1). \tag{5}$$

This equation can be solved by separation of variables to produce the most general possible exact solution if, and only if, either $\Phi' \equiv 0$ ($\Phi \equiv$ constant) or Φ is defined by $e^{2\Phi} = B_0 r^l$, i.e., $\Phi' = l/(2r)$, where B_0 is an arbitrary constant and l is fixed, first considered in Ref. [1].

In the former case $(\Phi' \equiv 0)$, we obtain $e^{2\Lambda} = 1/(1 - cr^{-(\omega+1)/\omega})$. Now if $\omega = -1/3$, then

$$e^{2\Lambda} = \frac{1}{1 - cr^2}, \tag{6}$$

which leads to a special case of the Friedmann-Lemaître-Robertson-Walker (FLRW) model. This outcome is consistent with the Friedmann equation

$$\frac{a''(t)}{a(t)} = -\frac{4\pi}{3}(\rho + 3p) = -\frac{4\pi}{3}(\rho + 3\omega\rho), \tag{7}$$

yielding $a''(t) = 0$ (no expansion) only if $\omega = -1/3$. Letting $k^2 = e^{2\Phi}$ and $c = K$, we obtain, after dividing by k^2 and rescaling, the FLRW line element with $a(t) \equiv 1$:

$$ds^2 = -dt^2 + \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{8}$$

In the present situation, we are more interested in the case $e^{2\Phi} = B_0 r^l$ for $l > 0$.

By Ref. [1],

$$ds^2 = -B_0 r^l dt^2 + \frac{dr^2}{\frac{\omega+1}{\omega+1+l} + Cr^{-(\omega+1+l)/\omega}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{9}$$

where C is a constant of integration. Keeping in mind the line element [2]

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{10}$$

where $m(r)$ is the total mass inside a sphere of radius r, we rewrite Eq. (9) as follows:

$$ds^2 = -B_0 r^l dt^2 + \frac{dr^2}{1 - 2r\left(\frac{1}{2}\right)\left[1 - \frac{\omega+1}{\omega+1+l} - Cr^{-(\omega+1+l)/\omega}\right]/r}$$

$$+\, r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (11)$$

We now see that

$$m(r) = r\left[\frac{1}{2}\left(1 - \frac{\omega+1}{\omega+1+l} - Cr^{-(\omega+1+l)/\omega}\right)\right], \qquad (12)$$

where $0 < \omega < 1$ (ordinary or dark matter). Since we are dealing with a region that is far removed from the galactic center, the last term in Eq. (12) becomes negligible. So $m(r)$ has the linear form $m(r) = ar$:

$$m(r) \approx \frac{1}{2} r\left(1 - \frac{\omega+1}{\omega+1+l}\right). \qquad (13)$$

As already noted, this form implies the existence of flat rotation curves and hence of dark matter. Returning to line element (9), recall that the exponent l has to be a constant in the exact solution obtained. So l has a natural interpretation thanks to the well-established model $B_0 r^l$, where $l = 2(v^{\phi})^2$ and v^{ϕ} is the tangential velocity. This model is obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(-e^{2\Phi(r)}\dot{t}^2 + e^{2\Lambda(r)}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right) \qquad (14)$$

by assuming a constant tangential velocity. (See, for example, Ref. [3]).

With the benefit of hindsight this qualitative result proves to be consistent with observation. Suppose m_1 is the mass of a star, v^{ϕ} its tangential velocity, and m_2 the total mass of everything else. (Here we need to use m_2 since $m(r)$ in Eq. (13) is only an approximation.) Then multiplying m_1 by the centripetal acceleration yields

$$m_1 \frac{(v^{\phi})^2}{r} = m_1 m_2 \frac{G}{r^2}. \qquad (15)$$

Since $G = 1$ and $(v^{\phi})^2 = \frac{1}{2} l$, we obtain $m_2 = \frac{1}{2} lr$. According to Ref. [4],

$l \approx 0.000001$. Since $m(r) < m_2$, Eq. (13) now leads to the requirement

$\omega > -0.000001$, which is met since $0 < \omega < 1$.

An example to illustrate Eq. (15) can be constructed from Table 1 of Ref. [5] on the Navarro-Frenk-White model. The entries correspond to a "virial'' radius ranging from 177 kpc for a dwarf galaxy to 3740 kpc for a rich galactic cluster. By choosing the large value of 342 kpc for r in Line 6, we may assume that the corresponding mass of $2.301 \times 10^{12} M_\odot$ corresponds to m_2 above. The given value of $v^\phi = 170.1\,\text{km/s}$ in the table agrees with the value computed from Eq. (15) (with $G = 6.67 \times 10^{-11}\,\text{N} \cdot \text{m}^2/\text{kg}^2$) to three significant figures.

3. Summary

By starting with the most general possible exact solution of the Einstein field equations given the perfect-fluid equation of state $p = \omega\rho$, it is proposed in this note that the following model for flat galactic rotation curves could have been predicted prior to 1970:

$$ds^2 = -B_0 r^l dt^2 + \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where

$$m(r) \approx \frac{1}{2} r\left(1 - \frac{\omega + 1}{\omega + 1 + l}\right), \quad 0 < \omega < 1,$$

and l is related to the tangential velocity.

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