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A NEW APPROACH TO THE EQUATIONS OF FREE FALL MOTION

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Abstract

In this paper, we obtained the equation of free fall motion to some special examples with completely different common methods in kinematics. Our original attention is on the mechanical energy relationship, $E = U + K$. The obtained motion constants in our approach are mechanical energy by weight which are not the same as common motion constants in kinematics equations. Our approach can make a new insight on the equations of free fall motion and it is truly impressive for students. Also we showed consistency between Newtonian and Lagrangian methods in finding the equation of motion of a particle sliding under the action of gravity on a curved path.

1. Introduction

In basic classical mechanics free fall motion is the fundamental argument and

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usually is deduced of Newton's equations. In basic text books kinematics methods are distinct from energy-work policies, for example, see [1]. In advanced mechanics for deducing equations of motion, we use Euler-Lagrange mechanics. Usually Lagrangian mechanics is difficult for its mathematics and hard being conceptions. It is not easy to understand its basic concepts including the principle of least action. Although there are some effective attempts to percept these concepts [2-6], also an interesting and very different inference can be found in [7].

In this paper, we tried to gain a new insight about equations of free fall motion without using Euler-Lagrange equation and Newton's. Our original relationship is:

$$E = U + K. \tag{1}$$

In this regard, E is the mechanical energy or the total energy, U potential energy and K kinetic energy. In all our examples friction is absent.

2. One Dimensional Equation of Motion

Particle is released under its own weight from a height (y) and total energy (E). Kinetic energy and Potential energy are as follows:

$$K = \frac{1}{2} m\dot{y}^2 \tag{2}$$

$$U = mgy. \tag{3}$$

We obtain directly from (1):

$$1 = \frac{U}{E} + \frac{K}{E}. \tag{4}$$

Here we define

$$\frac{U}{E} = \cos^2 \theta. \tag{5}$$

And

$$\frac{K}{E} = \sin^2 \theta. \tag{6}$$

In these equations θ is a time-dependent variable which creates a relationship between $\frac{U}{E}$ and $\frac{K}{E}$.

Substitute (2) and (3) into (5) and (6), we can write

$$\frac{mg}{E}y = \cos^2\theta, \tag{7}$$

$$\frac{m}{2E}\dot{y}^2 = \sin^2\theta. \tag{8}$$

The time derivative on (7) and comparing it with (8), we have

$$\cos\theta d\theta = g\sqrt{\frac{m}{2E}}dt. \tag{9}$$

After integration and putting it on (7) comes with the following equation of motion in the vertical direction

$$y = \frac{-1}{2}gt^2 + \frac{E}{mg}. \tag{10}$$

Our equation constant is the total energy of the particle by its weight although the constant of any other motion integrals are precisely different. As we know the constants of motion integrals are, during the process, unchanged [8].

Note that pointed motion constant in the (10) equals total energy by weight of particle. In this situation reader may think that this ratio is the primary height (because in this point $E = mgy + 0$) but we will see in next examples this ratio is going to be a new concept.

For example, a projectile which is thrown with primary velocity v_0 and angle α from horizon, its range and maximum height are

$$R_{\max} = \frac{2K}{mg}, \tag{11}$$

$$H_{\max} = \frac{K}{mg}, \tag{12}$$

K is the projectile primary kinetic energy.

And maximum range for a projectile which is thrown on inclined plane with primary velocity v_0 and angle α is

$$\frac{2k}{mg(1-\sin\alpha)}. \tag{13}$$

And finally maximum range for a projectile is

$$\frac{2K+U}{mg}. \tag{14}$$

According to (14), if the putter can give the projectile a certain amount of energy, he is twice as well off by giving the projectile kinetic rather than potential energy. But, of course, the putter cannot freely exchange these two types of energy as an extreme example, if the putter released projectile at ground level, his motion will be so awkward that he would be able to give the projectile only very little kinetic energy [9].

3. Two Dimensional Equation of Motion

Now we are searching for a main integral equation to obtain to the equations of two-dimensional motion.

In this state $U = mgy$ and $K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$.

Since $\dot{y} = \dot{x}f'(x)$, so with (5) and (6) we will have:

$$\frac{mg}{E}y = \cos^2\theta, \tag{15}$$

$$\frac{m}{2E}\left(\frac{\dot{y}^2}{f'(x)^2} + \dot{y}^2\right) = \sin^2\theta. \tag{16}$$

Again after derivative (15) and comparing with (16), we will get following integral equation.

$$\int \cos\theta d\theta = g\sqrt{\frac{m}{2E}}\int \frac{f'(x)}{1+f'(x)^2}dt. \tag{17}$$

The left hand side of (17) will give us the variable θ and in the right hand side $f'(x)$ which equals to ratio of $\frac{v_y}{v_x}$.

In every situation where $f'(x)$ is the function of time it has to be calculated in time-dependent integral (17).

In a situation where $f'(x)$ depends on other variable instead of time, we must achieve that variable as a function of time.

For example, consider a particle which is moving on an arbitrary trajectory as $y = f(x)$ so that the magnitude of velocity in every point of path is constant, v. In matter of fact, we imagine uniform motion on the curve path. Clearly the components of speed will be as follows

$$v_x = \frac{v}{\sqrt{1 + f'(x)^2}}, \tag{18}$$

$$v_y = \frac{\nu f'(x)}{\sqrt{1 + f'(x)^2}}. \tag{19}$$

Since $f(x)$ could be any arbitrary function in terms of x, therefore $f'(x)$ can also be a function of x but clearly it can be obtained as a function of time.

Now for investigating and performance of (17), we assumed that particle moved on the two-dimensional path where its ratio of speed components equals to

$$f'(x) = At + B, \tag{20}$$

where A and B are number constants.

Substituting (20) in (17) and after integration, the equation of motion will be obtained as follows

$$y = \frac{-1}{2} gt^2 + \frac{gB}{A} t - \frac{B^2 g}{2A^2} - \frac{g}{2A^2} + \frac{E}{mg}. \tag{21}$$

As we know for the projectile which is projected with primary velocity v_0 and angle

α than the horizon tall, we have

$$f'(x) = \frac{v_0 \sin \alpha - gt}{v_0 \cos \alpha}. \tag{22}$$

After comparing (22) with (20), the values *A* and *B* will be obtained

$$y = \frac{-1}{2} gt^2 + v_0 \sin \alpha t + \frac{E}{mg} - \frac{v_0^{\,2}}{2g}. \tag{23}$$

The last term in (23) is the primary kinetic energy and then motion constant in above equation equals to $\frac{E - K}{mg}$.

For the other examples, suppose a particle which is released on the inclined plane with angle α and we are going to obtain the equation of motion in the *y* axis direction. Due to $f'(x) = \tan \alpha$ and replacing it into (17), the equation of motion in vertical direction equals to

$$y = \frac{-1}{2} gt^2 \sin^2 \alpha + \frac{E}{mg}. \tag{24}$$

Note that here α is unchanged.

4. Motion on the Parabola

There are some researches to sliding particle's motion on the curved path [10-14]. However, it seems that they require to be investigated further. Now suppose a particle sliding along a curved path, parabola of $f(x) = x^2$, under the action of gravity like the before instances.

Obviously, $f'(x)$ in (17) equals to $\tan \alpha$ where α is time-dependent. In fact, with particle's move, the slope of tangential line on the path will change in every instant. In fact, (17) changes to following equation

$$\int \cos \theta d\theta = \frac{g}{2} \sqrt{\frac{m}{2E}} \int \sin 2\alpha \, dt.$$

Now we are sure α is the function of *t*. Searching for α is truly interesting and difficult.

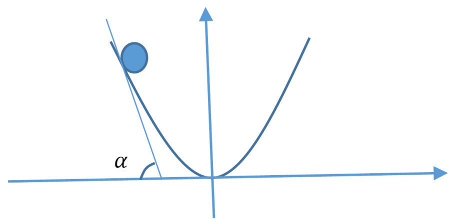

Figure 1. Particle is released on the parabola and slope of tangential line makes a new angle each second.

Since

$$f'(x) = \tan\alpha = 2x. \tag{25}$$

So

$$x = \frac{1}{2}\tan\alpha. \tag{26}$$

By derivative with respect to t from (26), we have

$$v_x = \frac{1}{2}(1 + \tan^2\alpha)\frac{d\alpha}{dt}. \tag{27}$$

Again with derivative with respect to t from $y = f(x) = \frac{1}{4}\tan^2\alpha$, we can write

$$v_y = \frac{1}{2}\tan\alpha(1 + \tan^2\alpha)\frac{d\alpha}{dt}. \tag{28}$$

And the magnitude of velocity of particle directed to the path is

$$v = \sqrt{{v_x}^2 + {v_y}^2} = \frac{1}{2\cos^2\alpha}\left(\frac{d\alpha}{dt}\right). \tag{29}$$

Newtown’s equation in the direction of the path is

$$m\frac{dv}{dt} = mg\sin\alpha. \tag{30}$$

By derivative from (29) and using (30), the differential equation due to this particle’s motion will be obtained

$$\frac{d^2\alpha}{dt^2} + 3\tan\alpha\left(\frac{d\alpha}{dt}\right)^2 - 2g\sin\alpha\cos^3\alpha = 0. \qquad (31)$$

Now we are going to rearrange this second order and nonlinear equation in terms of y. For this purpose, we have to differentiate (28) with respect to t then

$$\frac{\ddot{y}}{\sqrt{y}(1+4y)} - \frac{(1+12y)}{2y\sqrt{y}(1+4y)^2}\dot{y}^2 = \frac{d^2\alpha}{dt^2}. \qquad (32)$$

With replacing this equation in differential equation (31) and using again (28), we obtain final form

$$\ddot{y} - \frac{\dot{y}^2}{2y(1+4y)} + \frac{4gy}{1+4y} = 0. \qquad (33)$$

Differential equation (33) can directly be obtained from Euler-Lagrange equation. Considering gravitational potential energy of the particle in $U = mgy$ and its kinetic energy $K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ and suppose that $y = x^2$, the Lagrangian of the system is

$$L = K - U = \frac{1}{2}m\dot{y}^2\left(1 + \frac{1}{4y}\right) - mgy. \qquad (34)$$

By using Euler-Lagrange equation [6], that is,

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0, \qquad (35)$$

equation (33) will be obtained again.

Although (33) obtained simplicity method in Lagrangian view but in Newtonian perspective also leads to a nonlinear and second order differential equation.

To solve the deferential equation, firstly we will consider variable $\dot{y}$ as p, so that we can write

$$\ddot{y} = p\frac{dp}{dy}. \qquad (36)$$

And the differential equation will change into

$$pdp + \left(\frac{4gy}{1+4y} - \frac{p^2}{2y(1+4y)} \right) dy = 0. \tag{37}$$

Integral factor will be obtained easily and equals to $4 + \frac{1}{y}$ and then we will have a complete deferential equation with multiplying the integral factor by (37)

$$p\left(4 + \frac{1}{y}\right) dp + \left(4g - \frac{p^2}{2y^2}\right) dy = 0. \tag{38}$$

The answer of the above equation is

$$u(y,\ \dot{y}) = u(y,\ p) = k, \tag{39}$$

where k is a constant.

After some calculation function will be obtained as follows

$$u(y,\ p) = \frac{p^2}{2}\left(4 + \frac{1}{y}\right) + 4gy = k. \tag{40}$$

For calculating k, we use the only boundary condition in the problem which is given in Figure 2.

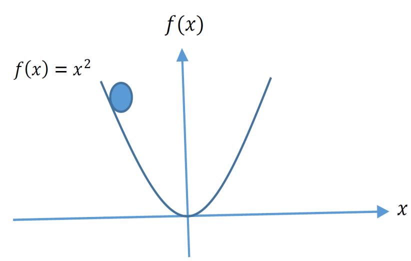

Figure 2. The only boundary condition being in origin, $x = 0,\ y = 0$ and $f'(x) = 0.$

We differentiate the familiar equation $\dot{y} = f'(x)\dot{x}$ with respect to t and we have

$$\ddot{y} = \ddot{x}f'(x) + \dot{x}^2 f''(x). \tag{41}$$

With attention to the only boundary condition as $(x = 0,\ y = 0)$ and $f'(x) = 0,$ so

$$\ddot{y} = 2\dot{x}^2. \tag{42}$$

It can be immediately concluded that

$$\lim_{y \to 0} \dot{y} \frac{d\dot{y}}{dy} = 2\dot{x}^2. \tag{43}$$

And then $k = 2\dot{x}^2$, so that equation (40) will be as follows

$$\frac{p^2}{2}\left(\frac{1+4y}{y}\right) + 4gy = 2\dot{x}^2. \tag{44}$$

With attention to $p = v_y$ and $\dot{x} = v_x$, we rearrange above equation

$$v_y = \frac{dy}{dt} = \sqrt{\frac{4yv_x^2 - 8gy^2}{1+4y}}. \tag{45}$$

Equation (45) will be an integral equation as follows

$$\int dt = \sqrt{\frac{-1}{8g}} \int \sqrt{\frac{1+4y}{\left(y - \frac{v_x^2}{4g}\right)\frac{v_x^4}{16g^2}}} dy. \tag{46}$$

To solve the integral, we change trigonometric variable

$$y - \frac{v_x^2}{4g} = \frac{v_x^2}{4g} \sec\theta. \tag{47}$$

By substituting in (46) and after slight calculation, we have

$$t = \frac{iv_x}{g\sqrt{8}} \int \sqrt{1 + \frac{g}{v_x^2} + \sec\theta} \sec\theta d\theta. \tag{48}$$

And final respond equals to

$$t = \frac{-2i}{\sqrt{8g}} E\left(1 + \frac{2v_x^2}{g}\right) - F\left(1 + \frac{2v_x^2}{g}\right). \tag{49}$$

$E(m)$ is the elliptic integral of the second kind with parameter $m = k^2$ and $F(m)$ is the elliptic integral of the first kind with parameter $m = k^2$.

5. Conclusion

We have tried to obtain the familiar equation of free fall motion in some common examples based on the relationship of the mechanical energy. Our method is different from any other common methods in kinematics, also the appeared motion constants are ratio of energy by weight of object which indicates that the primarily taken energy by object in the first position is not affectless in the equation of motion. Finally, we investigated the problem of releasing the object on the parabola and in this instance compatibility between Newtonian and Lagrangian methods is very clear. Especially, we showed that both of them lead to the same differential equation. In our opinion the simple shown procedure in this paper can make a better perception on some mechanical concepts.

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