

## A MODEL OF BPS BLACK HOLES IN A DISCRETE SPACE

NAHOMI KAN<sup>\*</sup>, KOICHIRO KOBAYASHI<sup>†</sup> and  
KIYOSHI SHIRAISHI<sup>††,\*</sup>

<sup>\*</sup>Gifu National College of Technology  
Motosu-shi, Gifu 501-0495  
Japan  
e-mail: [kan@gifu-nct.ac.jp](mailto:kan@gifu-nct.ac.jp)

<sup>†</sup>Center for Microelectronic System  
Kyushu Institute of Technology  
Iizuka-shi, Fukuoka 820-8502  
Japan  
e-mail: [koichiro\\_kobayashi@cms.kyutech.ac.jp](mailto:koichiro_kobayashi@cms.kyutech.ac.jp)

<sup>††</sup>Yamaguchi University  
Yamaguchi-shi, Yamaguchi 753-8512  
Japan  
e-mail: [shiraish@yamaguchi-u.ac.jp](mailto:shiraish@yamaguchi-u.ac.jp)

### Abstract

We examine a model of BPS black holes lying on a discrete extra space. The geometry is obtained from the discretization of the harmonic equation. We study the scattering amplitudes of two types of scalar fields, which correspond to fields in a bulk and on a brane. We conclude that the two types of scattering can be distinguished in the region of large transfer momentum.

### 1. Introduction

Extension of gravity theory is regarded as an important subject to study in

Keywords and phrases : black holes, higher dimensions, scattering.

<sup>\*</sup>Corresponding author

Received March 30, 2015

modern theoretical physics. Some extended models are expected to be relevant to the alternative of dark contents in the universe [1, 2]. Also in a microscopic perspective, modification of the Einstein gravity is motivated in the community of theoretical physicists; the theory with good quantum behavior and some natural explanation to the hierarchical scales in particle physics are eagerly pursued by many authors.

Now-a-days, the study of the models of gravity in higher dimensions, with and without higher-derivative terms in the action, has been a common topic in theoretical high-energy physics. Moreover, a broad range of possibilities is investigated, such as, scalar-tensor theory of gravity, vector-tensor theory, DBI-type, Lorentz-symmetry-breaking, non-local, and so on.

Massive gravity<sup>1</sup> is an interesting model for the modified gravity, because a massive graviton is a natural generalization in particle physics in popular sense but massive gravity turns out to have many difficulties as a quantum field theory. It is known that the construction of ghost-free massive gravity [5, 6] is naturally derived from bigravity, which has been studied for several decades [7, 8]. Interestingly, the generalization of bigravity may permit multigravity [9], and it is closely related with deconstruction of gravity. The dimensional deconstruction [10, 11] is an idea of making a higher-dimensional theory from the lower-dimensional copious fields. Thus, the dimensional deconstruction can be regarded as a modified and restricted version of the discretization of space, which assumes the minimal scale in the length scale. The idea of the smallest length in our universe has been considered for a long time as a solution of removing divergences in quantum field theory. So we have seen here, the various theories of gravity are mutually related.

It is essential to study the consequence of the generalized gravity theory with discreteness or other modifications at strong gravity, because it is known that the weak gravitational field limit is well-described by Einstein's general relativity. Therefore the solutions of the gravity theories which represent for gravitating localized objects are important theoretical arenas to investigate the feature of gravitation. Especially, the interaction with matter fields at strong gravity can be thoroughly studied if the exact solution of the space-time geometry is obtained.

In the present work, we will examine a simple model of a BPS black hole with a

---

<sup>1</sup>For reviews, see [3, 4].

discrete space. In general, the object possessing the BPS relation in its mass and charges is governed by simple equation of motion, and is usually motivated by string theory and theories with supersymmetries. The BPS equation considered here is the Laplace equation, thus, the discretization of the differential equation can be done rather in a straightforward way. In this paper, we introduce the graph Laplacian to perform the discretization. Therefore, the extension to the general structures of discrete spaces associated with generic graphs will be possible, though only the simplest case is treated in the present paper.

The plan of the present paper is as follows. In Section 2, we first review BPS black hole solutions<sup>2</sup> in the Einstein-Maxwell-dilaton theory. Subsequently, we introduce the graph Laplacian to discretize the BPS equation and show its solution in the simplest case. In Section 3, we study the scattering amplitudes of scalar fields with the BPS black hole with a discretized extra space. We treat them in the Born approximation in the present paper. We consider two types of scalar fields, one is obtained from the discretization of the continuum theory, another is the field living in one site of the discrete space. We concentrate ourselves on finding the way how we can ‘see’ the black hole by different kinds of scalar fields. Section 5 is devoted to summary and outlook.

## 2. BPS Black Holes and Discretization

In this section, we review the simplest system allowing a BPS solution, the Einstein-Maxwell-dilaton theory. The action for the model in  $D$ -dimensional (continuous) spacetime is given by [12, 13, 14]

$$S = \int d^D x \frac{\sqrt{-g}}{16\pi} \left[ R - \frac{4}{D-2} (\nabla\phi)^2 - e^{-4\alpha\phi/(D-2)} F^2 \right], \quad (1)$$

where  $R$  denotes the scalar curvature,  $\phi$  stands for the dilaton field, and the field strength is defined as  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  with an abelian gauge field  $A_\mu$ .

In this action,  $\alpha$  is the dilaton coupling. The effective massless field theory of string theory can be obtained if one set as  $\alpha = 1$ . Then, the appropriate scaling of the metric yields the following ‘stringy’ action:

---

<sup>2</sup>Strictly speaking, the solution obtained in the BPS limit has the singularity in the Einstein frame, except for the Reissner-Nordström solution ( $\alpha = 0$ ).

$$\tilde{S} = \int d^D x \frac{\sqrt{-\tilde{g}}}{16\pi} e^{-2\phi} [\tilde{R} + 4(\nabla\phi)^2 - \tilde{F}^2] \quad (2)$$

with  $\tilde{g}_{\mu\nu} = e^{\frac{4}{D-2}\phi} g_{\mu\nu}$ . Now, we turn to use the original metric (1) in the following discussion.

The static BPS solution can be derived, with the following ansätze:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -V^{-2(D-3)/(D-3+\alpha^2)} dt^2 + V^{2/(D-3+\alpha^2)} dx^2, \quad (3)$$

$$e^{-4\alpha\phi/(D-2)} = V^{2\alpha^2/(D-3+\alpha^2)}, \quad (4)$$

$$A_\mu dx^\mu = \sqrt{\frac{D-2}{2(D-3+\alpha^2)}} (1 - V^{-1}) dt, \quad (5)$$

as the solution of the Laplace equation for  $V(x^i)$ :

$$\partial^2 V = 0, \quad (6)$$

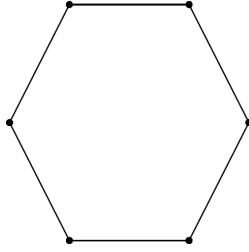
where the Laplacian is

$$\partial^2 \equiv \partial_i \partial^i, \quad (i = 1, \dots, D-1). \quad (7)$$

Here we ‘discretize’ the equation (6) by replacing a part of the Laplacian with a certain graph Laplacian. We adopt

$$\partial^2 \rightarrow \sum_{i=1}^d \partial_i \partial^i - a^{-2} \Delta(G), \quad (8)$$

where  $a$  is a scale of length. The graph Laplacian  $\Delta(G)$  has been introduced in spectral graph theory [15-18].



**Figure 1.** A cycle graph,  $C_6$ .

A graph consists of vertices (or sites) and edges which link two vertices. For example, a cycle graph  $C_N$  has  $N$  vertices and  $N$  edges connecting vertices circularly (Figure 1). A matrix is defined according to the manner of connections of edges to vertices, and is called a *graph Laplacian*. For example, the graph Laplacian of  $C_6$  is written as

$$\Delta(C_6) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \quad (9)$$

The equation characterizing the eigensystem of the matrix is

$$\Delta(G)v^{(\ell)} = \lambda^{(\ell)}v^{(\ell)}, \quad (10)$$

where  $\lambda^{(\ell)}$  is an eigenvalue of  $\Delta(G)$  and  $v^{(\ell)}$  is an eigenvector belonging to the eigenvalue.

For  $G = C_N$ , one can find that the eigenvalues

$$\lambda^{(\ell)} = 4 \sin^2 \frac{\pi \ell}{N} \quad (11)$$

and the eigenvectors

$$v^{(\ell)} = (v_0^{(\ell)}, v_1^{(\ell)}, \dots, v_k^{(\ell)}, \dots, v_{N-1}^{(\ell)})^T$$

$$\frac{1}{\sqrt{N}} \left( 1, e^{i \frac{2\pi \ell}{N}}, \dots, e^{i \frac{2\pi \ell}{N} k}, \dots, e^{i \frac{2\pi \ell}{N} (N-1)} \right)^T, \quad (12)$$

where  $\ell = 0, 1, \dots, N-1$ . Note that the normalization of inner products can be fixed as

$$v^{(\ell)*} \cdot v^{(\ell')} = \delta_{\ell \ell'}. \quad (13)$$

Now we turn to the (partial) discretization of the Laplace equation. We restrict ourselves on the case with the cycle graph  $C_N$ , hereafter. Note that the similar

discussion for general graphs is possible. The function  $V$  should be interpreted as the functions associated with vertices of a graph. By using the eigenvectors, we can expand

$$V_k = \sum_{\ell=0}^{N-1} V^{(\ell)} v_k^{(\ell)}, \quad (14)$$

where  $N$  is the number of eigenvectors,  $k$  stands for the  $k$ -th vertex and  $V^{(\ell)}$  is the function of  $x^i$ ,  $i = 1, \dots, d$ . Assuming a ‘localized’ source, at the zero-th vertex, the equation should read

$$\left[ \sum_{i=1}^d \partial_i \partial^i - a^{-2} \Delta(C_N) \right] V_k = -4\pi\mu N \delta^d(x^i) \delta_{k0}. \quad (15)$$

We find the solution for  $d = 3$ :

$$V_k(r) = 1 + \frac{\mu}{r} \sum_{\ell=0}^{N-1} \exp\left[-2\left|\sin \frac{\pi\ell}{N}\right| \frac{r}{a}\right] e^{i\frac{2\pi\ell}{N}k}, \quad (16)$$

which seems to be a sum of the Newtonian potential and the Yukawa-type potentials.

Taking the limit of a large number of vertices, such that  $N \rightarrow \infty$  and a small discretized scale  $a \rightarrow 0$  while  $L \equiv Na$  is constant, with introducing a continuous parameter  $y \equiv ka$  ( $0 \leq y < L$ ), we obtain

$$\begin{aligned} \sum_{\ell=0}^{N-1} \exp\left[-2\left|\sin \frac{\pi\ell}{N}\right| \frac{r}{a}\right] e^{i\frac{2\pi\ell}{N}k} &\rightarrow \sum_{\ell=-\infty}^{\infty} \exp\left[-\frac{2\pi|\ell|}{N} \frac{r}{a}\right] e^{i\frac{2\pi\ell}{N}k} \\ &= \frac{\sinh(2\pi r/L)}{\cosh(2\pi r/L) - \cos(2\pi y/L)}. \end{aligned} \quad (17)$$

This expression has appeared when we considered the BPS black holes in the Kaluza-Klein compactification on  $S^1$  with the circumference  $L$  [19].

Incidentally, the solution (16) can be expressed by the infinite sum as

$$\sum_{\ell=0}^{N-1} \exp\left[-2\left|\sin \frac{\pi\ell}{N}\right| \frac{r}{a}\right] e^{i\frac{2\pi\ell}{N}k}$$

$$\begin{aligned}
 &= \sum_{q=-\infty}^{\infty} \left\{ \frac{Nr}{\pi a} \left[ (Nq - k)^2 - \frac{1}{4} \right]^{-1} {}_1F_2(1; 3/2 - (Nq - k), 3/2 + (Nq - k); r^2 / a^2) \right. \\
 &\quad \left. + \frac{N}{\cos[(Nq - k)\pi]} I_{2(Nq - k)}(2r/a) \right\}. \tag{18}
 \end{aligned}$$

Every part in the parentheses has the limit

$$\{\dots\} \xrightarrow{N \rightarrow \infty, Na=L} \frac{Lr}{\pi} \frac{1}{r^2 + (Lq - y)^2}, \tag{19}$$

which is the Green's function in four-dimensional space with mirror sources. Thus every part in the parentheses corresponds to the Green's function in three-dimensional space and one-dimensional infinite lattice.

### 3. Scattering Amplitudes of Scalar Waves

Scattering by black holes has been studied in the literature, such as [20]. In this section, we treat the scattering only in the simple way, since the model we consider here is still a toy model. We consider scattering of scalar wave in four dimensional space-time, i.e.,  $d = 3$ . Suppose that the wave equation is assumed as

$$\sum_{i=1}^3 \partial_i \partial_i \psi + p^2 \psi - U(r) \psi = 0, \tag{20}$$

where  $U(r)$  is an effective potential for scattering and  $p$  stands for the momentum of the incident wave.

It is well known [21] that the Born approximation leads to the form of the following scattering amplitude  $f(\theta)$ , where  $\theta$  denotes the scattering angle,

$$f(\theta) = -\frac{1}{q} \int_0^\infty r U(r) \sin q r dr \tag{21}$$

with the transfer momentum

$$q = 2p \sin \frac{\theta}{2} \tag{22}$$

and  $p$  is the wave number of the incident wave. The scattering cross-section is simply

given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2. \quad (23)$$

We will consider two types of scalar fields and how the black hole described by the solution (16) can be seen by the waves in the Born approximation.

### 3.1. Massless scalar field in the ‘bulk’

We first consider the scalar field originally defined in  $D$ -dimensional space-time and define its discretized version, which corresponds to the scalar plane wave spreading in the bulk space. Thus, we shortly call this type of scalar field as the ‘bulk’ scalar.

The wave function of the massless scalar in continuum  $D$ -dimensional space-time is written as

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \psi) = 0. \quad (24)$$

If the background geometry with the solution (16) is substituted and monochromatic wave  $\psi \propto e^{-i\omega t}$  is presumed, the wave equation takes the form

$$\partial^2 \psi + V^{2(D-2)/(D-3+\alpha^2)} \omega^2 \psi = 0. \quad (25)$$

Note that for the massless field,  $\omega = p$ . For we consider a scalar field in the bulk now, we adopt the lowest eigenstate as the  $d+1$ -dimensional massless scalar. We assume, that is,

$$\psi \rightarrow \psi^{(0)}, \quad \Delta(G)\psi^{(0)} \approx 0. \quad (26)$$

Therefore the wave equation becomes

$$\sum_{i=1}^d \partial_i \partial_i \psi_k^{(0)} + V_k^{2(D-2)/(D-3+\alpha^2)} \omega^2 \psi_k^{(0)} = 0. \quad (27)$$

To use the Born approximation for  $d = 3$ , we should use the trace of the matrix element of state vector from  $k = 0$  to  $k = N - 1$ . Thus, the scattering amplitude is given by



$$f(\theta) = -\frac{1}{q} \int_0^\infty r U(r) \sin q r dr \quad (28)$$

with

$$U(r) = -\frac{\omega^2}{N} \sum_{k=0}^{N-1} (V_k^{2(D-2)/(D-3+\alpha^2)} - 1). \quad (29)$$

One can find a special case. If we consider the case with the dilaton coupling  $\alpha = 1$ , then we find

$$U(r) = -\frac{\omega^2}{N} \sum_{k=0}^{N-1} (V_k^2 - 1), \quad (30)$$

and this is independent of the dimensionality  $D$ . Substituting the solution of  $V_k$  for  $C_N$  obtained in the previous section, we get

$$-\frac{1}{\omega^2} U(r) = \frac{2\mu}{r} + \frac{\mu^2}{r^2} \sum_{\ell=0}^{N-1} \exp\left[-4\left|\sin \frac{\pi\ell}{N}\right| \frac{r}{a}\right], \quad (31)$$

and this leads to the following scattering amplitude:

$$\frac{1}{\omega^2} f(\theta) = \frac{2\mu}{q^2} + \frac{\mu^2}{q} \sum_{\ell=0}^{N-1} \arctan \frac{aq}{4\left|\sin \frac{\pi\ell}{N}\right|}. \quad (32)$$

The amplitude as the function of the transfer momentum in this case is shown in Figure 2 for  $N = 6$  and Figure 3 for  $N = 30$ .

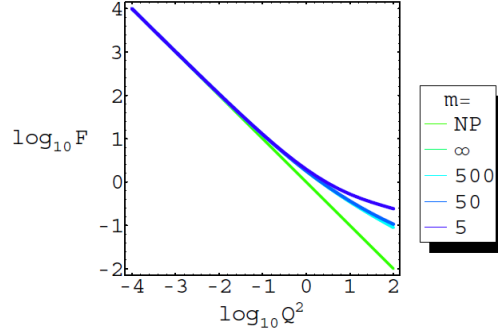
Here we define normalized quantities  $F \equiv \left| \frac{f(q)}{2\mu^3\omega^2} \right|$  and  $Q^2 \equiv \mu^2 q^2$ . In each

figure, curves for several different values of  $m \equiv \mu/a$  are indicated. The line indicated as ‘NP’ represents for the amplitude by the pure Newton potential for a reference case. Note the variables defined here will be used throughout the present paper.

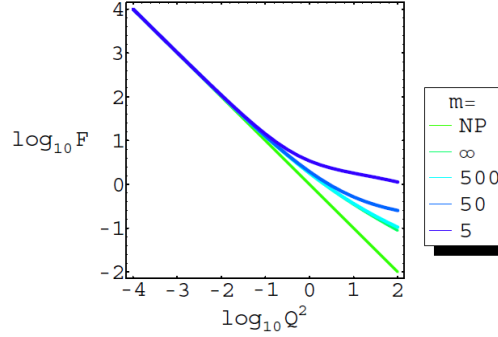
Since the constant  $\mu$ , indicates the size of the ‘black hole’, the next leading contribution to the Newtonian potential can be detected at large  $Q$ . When the scale of

discreteness (or the minimal length)  $a$  is sufficiently small compared with  $\mu$ , the dependence on  $N$  becomes small.

The limit of large  $N$  and small  $a$  should yield the case with a compactified space  $S^1$  in the continuum theory, in this case with  $C_N$ . The amplitude for a fixed  $\mu/(Na)$  is shown in Figure 4 for  $\mu/(Na) = 1$  and Figure 5 for  $\mu/(Na) = 10$ . In each figure, curves for  $N = 3, 6, 30$  are plotted.



**Figure 2.** The scattering amplitude of the ‘bulk’ scalar for  $N = 6$ . From top to the bottom on the right-hand side of the curves correspond to  $m = 5, 50, 500, \infty$  and the case with pure Newton potential, respectively.



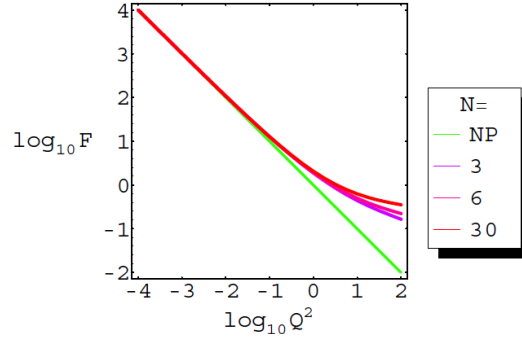
**Figure 3.** The scattering amplitude of the ‘bulk’ scalar for  $N = 30$ . From top to the bottom on the right-hand side of the curves correspond to  $m = 5, 50, 500, \infty$  and the case with pure Newton potential, respectively.

We can find that if the scale of the ‘extra dimension’  $Na$  is small compared with the black hole scale  $\mu$ , the discreteness of the space is difficult to be detected.

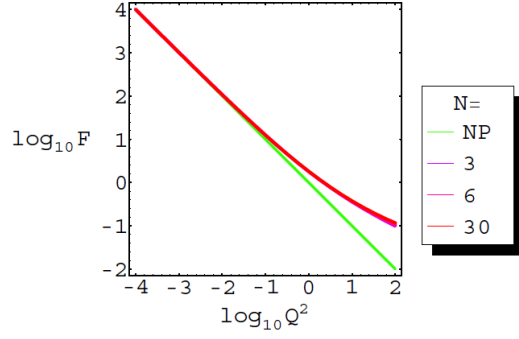
### 3.2. Massless scalar field confined on the ‘brane’

Next we consider the scalar field living in the single vertex at which the source of the BPS black hole located. This is a mimicker of the field confined on a brane. Thus, we may abbreviatedly call this type of scalar field as the ‘brane’ scalar. The corresponding brane is identified with  $k = 0$  vertex. The wave equation for the scalar field  $\Psi_0$  at the zeroth vertex is

$$\frac{1}{\sqrt{-q}} \partial_a (\sqrt{-q} q^{ab} \partial_b \Psi_0) = 0, \quad (33)$$



**Figure 4.** The scattering amplitude of the ‘bulk’ scalar for  $\mu/(Na) = 1$ . From top to the bottom on the right-hand side of the curves correspond to  $N = 30, 6, 3$  and the case with pure Newton potential, respectively.



**Figure 5.** The scattering amplitude of the ‘bulk’ scalar for  $\mu/(Na) = 10$ . From top to the bottom on the right-hand side of the curves correspond to  $N = 30, 6, 3$  and the case with pure Newton potential, respectively.

where  $q_{ab}$  ( $a, b = 0, 1, \dots, d$ ) denotes the  $d + 1$ -dimensional metric defined through

$$q_{ab} dx^a dx^b = -V_0^{-2(D-3)/(D-3+\alpha^2)} dt^2 + V_0^{2/(D-3+\alpha^2)} \sum_{i=1}^d dx^{i^2}. \quad (34)$$

Assuming  $\Psi_0 \propto e^{-i\omega t}$  again, we get the following wave equation:

$$\sum_{i=1}^d \partial_i V_0^{(d+1-D)/(D-3+\alpha^2)} \partial_i \Psi_0 + V_0^{(d+D-3)/(D-3+\alpha^2)} \omega^2 \Psi_0 = 0. \quad (35)$$

Further we rewrite the equation by using the new variable

$$\Psi_0 = V_0^{-\frac{d+1-D}{2(D-3+\alpha^2)}} \psi_0, \quad (36)$$

as

$$\begin{aligned} & \sum_{i=1}^d \partial_i \partial_i \psi_0 + V_0^{2(D-2)/(D-3+\alpha^2)} \omega^2 \psi_0 \\ & - \frac{d+1-D}{D-3+\alpha^2} \left[ \frac{d-3D+7-2\alpha^2}{2(D-3+\alpha^2)} \sum_{i=1}^d \frac{\partial_i V_0}{V_0} \frac{\partial_i V_0}{V_0} + \sum_{i=1}^d \frac{\partial_i \partial_i V_0}{V_0} \right] \psi_0 = 0. \end{aligned} \quad (3.7)$$

Although this equation depends on the number of spatial dimensions  $d$  as well as the total dimensionality  $D$ , the last term can be neglected compared with the second term if we consider sufficiently high-energy scattering.

For  $d = 3$  and  $\alpha = 1$ , the wave equation reads at high energy as

$$\sum_{i=1}^3 \partial_i \partial_i \psi_0 + V_0^2 \omega^2 \psi_0 \approx 0, \quad (38)$$

where

$$V_0(r) = 1 + \frac{\mu}{r} \sum_{\ell=0}^{N-1} \exp \left[ -2 \left| \sin \frac{\pi \ell}{N} \right| \frac{r}{a} \right]. \quad (39)$$

Therefore the effective potential is given by

$$U(r) = -\omega^2 (V_0^2 - 1). \quad (40)$$

By the Born approximation, we obtain the following scattering amplitude:

$$\begin{aligned} \frac{1}{\omega^2} f(\theta) = & 2\mu \frac{Na^2[(\sqrt{4+a^2q^2}+aq)^N + (\sqrt{4+a^2q^2}-aq)^N]}{\sqrt{4a^2q^2+a^4q^4}[(\sqrt{4+a^2q^2}+aq)^N - (\sqrt{4+a^2q^2}-aq)^N]} \\ & + \frac{\mu^2}{q} \sum_{\ell_1=0}^{N-1} \sum_{\ell_2=0}^{N-1} \arctan \frac{aq}{2\left(\left|\sin \frac{\pi \ell_1}{N}\right| + \left|\sin \frac{\pi \ell_2}{N}\right|\right)}. \end{aligned} \quad (41)$$

The amplitude in this case is shown in Figure 6 for  $N = 6$  and in Figure 7 for  $N = 30$ . For the ‘brane’ scalar, the dependence on  $m = \mu/a$  is large for large  $N$ .

The amplitude for fixed  $\mu/(Na)$  is shown in Figure 8 for  $\mu/(Na) = 1$  and in Figure 9 for  $\mu/(Na) = 10$ . Even if the scale of compactification  $\mu/(Na)$  is large, the dependence on  $N$  is not so small.

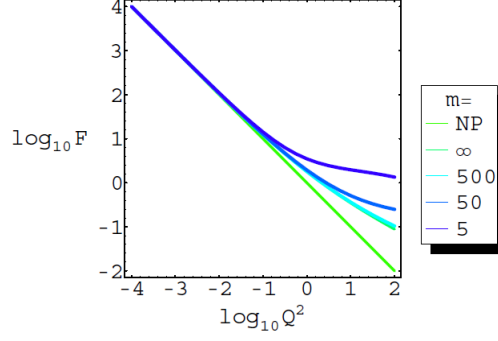
Since the ‘bulk’ scalar couples only to the Newtonian potential at the leading order, the dependence on  $N$  is rather small. This is because  $V^{(0)} \propto \mu/r$  and the incident wave  $\psi \propto \psi^{(0)}$ .

On the other hand, the ‘brane’ scalar couples to every mode, thus the amplitude is sensitive to all the ratios of variables.

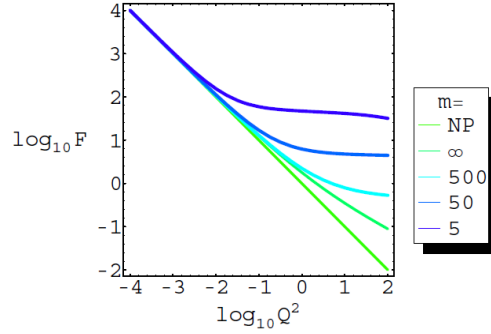
#### 4. Summary and Outlook

To summarize: We consider the discretization of the BPS equation and obtain a solution in a simple case, which has a continuum limit of  $S^1$  compactification. The solution in the present paper has three length scales: the radius of the black hole  $\approx \mu$ , the discretization scale  $a$ , and the scale of the ‘extra dimension’  $Na$ .

The scattering of scalar fields has been studied. The dependence on the ratio of the variables differs by the type of scalar fields, the ‘bulk’ scalar and the ‘brane’ scalar. The ‘bulk’ scalar of the Kaluza-Klein zero mode couples to  $1/r$  potential at the lowest order in  $\mu$ , the dependence of amplitude on  $N$  is rather small. On the other hand, the ‘brane’ scalar couples to all the components of the potential from the black hole, therefore the amplitude has large dependence on  $N$ .



**Figure 6.** The amplitude of the ‘brane’ scalar for  $N = 6$ . From top to the bottom on the right-hand side of the curves correspond to  $m = 5, 50, 500, \infty$  and the case with pure Newton potential, respectively.

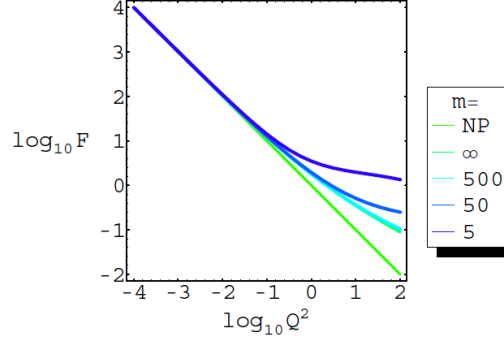


**Figure 7.** The amplitude of the ‘brane’ scalar for  $N = 30$ . From top to the bottom on the right-hand side of the curves correspond to  $m = 5, 50, 500, \infty$  and the case with pure Newton potential, respectively.

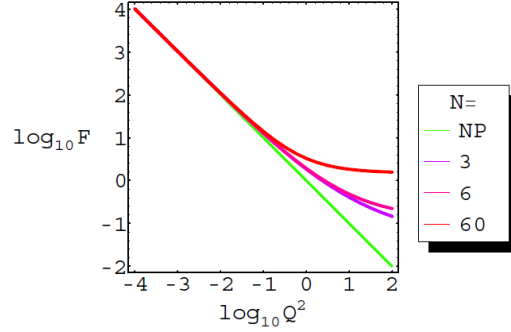
The further study and straightforward extensions of the present work are expected as follows. The higher-order in the approximation or numerical derivation of the scattering amplitude should be checked. The solution describing multi-black holes can also be obtained and the scattering by the multi-black holes can be calculated. The use of other graphs than  $C_N$  is of importance, such as a path graph  $P_N$ , which imitates  $\sim S^1/Z_2$  in the continuum limit. The graph with vertex weights is analogous to a warped space and is worth examining. The discretization using disconnected graphs seems to be a possible non-trivial extension.

We also notify that the scattering by the stringy BPS black hole, in the case with

$\alpha = 1$ , is independent of the spatial dimensionality. This is important, if we consider generalization of the model using the complex graphs. The graph structure has, in general, no continuum limit in a naive sense as the case considered in the present paper ( $C_N \rightarrow S^1$ ). As the field theory on fractal graphs have been studied [22, 23], the gravity on fractal graphs is an exciting subject to study. The fractal has a unusual dimension, or in some cases, no uniquely-defined dimension. The stringy case or special choice of the coupling is substantial in the study of theory with the fractal (graph).



**Figure 8.** The amplitude of the ‘brane’ scalar for  $\mu/(Na) = 1$ . From top to the bottom on the right-hand side of the curves correspond to  $N = 30, 6, 3$  and the case with pure Newton potential, respectively.



**Figure 9.** The amplitude of the ‘brane’ scalar for  $\mu/(Na) = 10$ . From top to the bottom on the right-hand side of the curves correspond to  $N = 30, 6, 3$  and the case with pure Newton potential, respectively.

The present approach is based on the discretization of the equation of motion,

thus the action of the complete theory has not been considered yet. In other words, the discretization in our approach is only valid for the case with a special BPS relation among mass and charges. Although the investigation into the general case is important, the BPS case may be a special point in the ‘running’ couplings and to study the deviation from the point may be effective at some energy scale.

Finally, we notify that a discrete object with a certain symmetry is interesting from a mathematical point of view, and the model of magnetic monopole has been considered recently [24]. Anyway, investigation on the possible substructure of our space-time should be continued with various approaches.

### References

- [1] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, *Phys. Rep.* 513 (2012), 1.
- [2] S. Capozziello and M. De Laurentis, *Phys. Rep.* 509 (2011), 167.
- [3] K. Hinterbichler, *Rev. Mod. Phys.* 84 (2012), 671.
- [4] C. de Rham, *Living Rev. Rel.* 17 (2014), 7.
- [5] C. de Rham, G. Gabadadze and A. J. Tolley, *Phys. Rev. Lett.* 106 (2011), 231101.
- [6] S. F. Hassan, R. Rosen and A. Schmidt-May, *JHEP* 1202 (2012), 026.
- [7] C. J. Isham, A. Salam and J. Strathdee, *Phys. Rev. D* 3 (1971), 867; A. Salam and J. Strathdee, *Phys. Rev. D* 16 (1977), 2668; A. Salam and J. Strathdee, *Phys. Lett. B* 67 (1977), 429; C. J. Isham and D. Storey, *Phys. Rev. D* 18 (1978), 1047.
- [8] T. Damour and I. I. Kogan, *Phys. Rev. D* 66 (2002), 104024; D. Blas, C. Deffayet and J. Garriga, *Class. Quant. Grav.* 23 (2006), 1697; D. Blas, C. Deffayet and J. Garriga, *Phys. Rev. D* 76 (2007), 104036; D. Blas, *Int. J. Theor. Phys.* 46 (2007), 2258; Z. Berezhiani, D. Comelli, F. Nesti and L. Pilo, *Phys. Rev. Lett.* 99 (2007), 131101; N. Rossi, *Eur. Phys. J. ST* 163 (2008), 291; Z. Berezhiani, D. Comelli, F. Nesti and L. Pilo, *JHEP* 0807 (2008), 130; Z. Berezhiani, F. Nesti, L. Pilo and N. Rossi, *JHEP* 0907 (2009), 083; Z. Berezhiani, L. Pilo and N. Rossi, *Eur. Phys. J. C* 70 (2010), 305; M. Bañados, A. Gomberoff and M. Pino, *Phys. Rev. D* 84 (2011), 104028.
- [9] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, *Ann. Phys.* 305 (2003), 96; N. Arkani-Hamed and M. D. Schwartz, *Phys. Rev. D* 69 (2004), 104001; M. D. Schwartz, *Phys. Rev. D* 68 (2003), 024029; G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, *Mod. Phys. Lett. A* 19 (2004), 1435; S. Nojiri and S. D. Odintsov, *Phys. Lett. B* 590 (2004), 295; F. Bauer, T. Hallgren and G. Seidl, *Nucl. Phys. B* 781 (2007), 32; T. Hanada, K. Kobayashi, K. Shinoda and K. Shiraishi, *Class. Quant.*



- Grav. 27 (2010), 225010; K. Hinterbichler and R. A. Rosen, JHEP 1207 (2012), 047; S. A. Duplij and A. T. Kotvytskiy, Theor. Math. Phys. 177 (2013), 1400; N. Tamanini, E. N. Saridakis and T. S. Koivisto, JCAP 1402 (2014), 015; C. de Rham, A. Matas and A. J. Tolley, Class. Quant. Grav. 31 (2014), 025004; J. Noller, J. H. C. Scargill and P. G. Ferreira, JCAP 1402 (2014), 007.
- [10] N. Arkani-Hamed, A. G. Cohen and H. Georgi, Phys. Rev. Lett. 86 (2001), 4757.
- [11] C. T. Hill, S. Pokorski and J. Wang, Phys. Rev. D 64 (2001), 105005.
- [12] G. W. Gibbons and K. Maeda, Nucl. Phys. B 298 (1988), 741.
- [13] D. Garfinkle, G. T. Horowitz and A. Strominger, Phys. Rev. D 43 (1991), 3140.
- [14] K. Shiraishi, J. Math. Phys. 34 (1993), 1480.
- [15] A. E. Brouwer and W. H. Haemers, Spectra of Graphs, Springer, New York, 2012.
- [16] C. Godsil and G. Royle, Algebraic Graph Theory, Springer, New York, 2001.
- [17] D. Cvetković, P. Rowlinson and S. Simić, An Introduction to the Theory of Graph Spectra (London Mathematical Society Student Texts 75), Cambridge University Press, 2010.
- [18] N. Kan and K. Shiraishi, J. Math. Phys. 46 (2005), 112301.
- [19] K. Shiraishi, Soryushiron Kenkyu (Kyoto) 93 (1996), 310.
- [20] J. A. H. Futterman, F. A. Handler and R. A. Matzner, Scattering from Black Holes, Cambridge University Press, 1988.
- [21] A. Messiah, Quantum Mechanics, Dover, 2014.
- [22] C. T. Hill, Phys. Rev. D 67 (2003), 085004.
- [23] C. T. Hill, arXiv:hep-th/0303267.
- [24] G. M. Kemp, Discrete analogues of Dirac's magnetic monopole and binary polyhedral groups, arXiv:1310.0055 [math-ph].