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THE FRINGING CAPACITANCE OF AN INCLINED PLATE CAPACITOR

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Abstract

The fringing capacitance of an inclined plate capacitor is investigated. The conformal transformations mapped the electric field into a rectangular region. The calculation is aimed at the general case. Any location of the electrode plate is included. It stands out that making use of separating method in solving Laplace equation to cope this problem is an approximate way under some restriction conditions.

1. Introduction

Capacitor is an important kind of electrical elements widely used in engineering. The most common case of capacitors is known as parallel plate capacitor which consists of two conducting plates parallel to each other with the same size and the fringing effect of the capacitor is not considered. When an electrode plate is slightly

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inclined and not parallel to another plane anymore, the fringing capacitance goes up. The chief aim of this paper is to calculate the fringing capacitance of the inclined plate capacitor in general case where the geometrical sizes of the plates are not the same. The two plates have the same size is treated as a special instance. Some relative approximate solution is displayed and the application conditions are analyzed.

2. Capacitance

A capacitor consists of two non-parallel conducting plates with sufficient longitudinal length. Its cross section in z-plane is sketched in Figure 1. The prolonged lines of the plate AB and CD intersect at point O. Denote $\angle AOC = \varphi$ and suppose the voltage across the two plates is V.

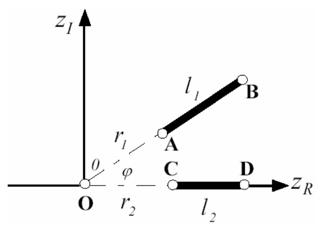

Figure 1. The cross section of inclined plate capacitor in z-plane.

Employing the conformal transformation [1]

$$t = \frac{2z^{\frac{\pi}{\varphi}} + (r_1 + l_1)^{\frac{\pi}{\varphi}} - (r_2 + l_2)^{\frac{\pi}{\varphi}}}{(r_1 + l_1)^{\frac{\pi}{\varphi}} + (r_2 + l_2)^{\frac{\pi}{\varphi}}}, \tag{1}$$

the electric field region in the z-plane is mapped onto the upper t-plane of Figure 2.

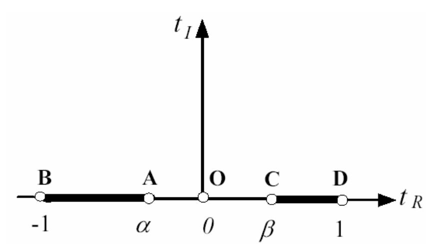

Figure 2. The t plane.

The parameters α and β are as following:

$$\alpha = \frac{-2r_1^{\frac{\pi}{\varphi}} + (r_1 + l_1)^{\frac{\pi}{\varphi}} - (r_2 + l_2)^{\frac{\pi}{\varphi}}}{(r_1 + l_1)^{\frac{\pi}{\varphi}} + (r_2 + l_2)^{\frac{\pi}{\varphi}}}, \tag{2}$$

$$\beta = \frac{2r_2^{\frac{\pi}{\varphi}} + (r_1 + l_1)^{\frac{\pi}{\varphi}} - (r_2 + l_2)^{\frac{\pi}{\varphi}}}{(r_1 + l_1)^{\frac{\pi}{\varphi}} + (r_2 + l_2)^{\frac{\pi}{\varphi}}}. \tag{3}$$

Referring Figure 3, the upper *t*-plane is transformed into the interior of the rectangle *ABDC* in the *u*-plane by

$$t = \alpha + \frac{(1-\alpha)(1+\alpha)}{2sn^2(u,\, k) + \alpha - 1}, \tag{4}$$

where $sn(u,\, k)$ is Jacobi elliptic function [2] and the modulus *k* is

$$k = \sqrt{\frac{2(\beta - \alpha)}{(1-\alpha)(1+\beta)}}. \tag{3}$$

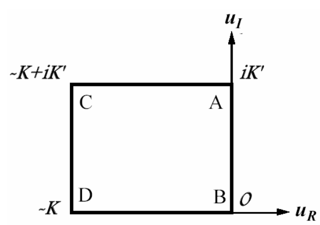

Figure 3. The *u* plane.

Using the two conformal transformations, a region confined by the rectangle *ABDC* in the *u*-plane filled with a uniform field is achieved. Consequently, the capacitance per unit longitudinal length of the plate is [3]

$$C = \varepsilon_0 \frac{K'(k)}{K(k)}, \tag{6}$$

where $K(k)$ is the complete elliptic integral of the first kind, and

$$K'(k) = K(k'). \tag{7}$$

The k' indicates the complementary modulus of *k*, e.g.,

$$k' = \sqrt{\frac{(1+\alpha)(1-\beta)}{(1-\alpha)(1+\beta)}}. \tag{8}$$

3. The Inner Capacitance

If the fringing effect is neglected, we only consider the inner capacitance. Laplace equation $\nabla^2\Phi = 0$ in the electric field could be solved with the method of separating variables in cylindrical coordinates [4]. That implies

$$\frac{1}{\rho^2}\frac{d^2\Phi}{d\phi^2} = 0, \tag{9}$$

so

$$\Phi = A\phi + B. \tag{10}$$

Making use of the boundary conditions

$$\Phi\Big|_{\phi=0} = 0; \quad \Phi\Big|_{\phi=\varphi} = V, \tag{11}$$

we have

$$A = \frac{V}{\varphi}; \quad B = 0. \tag{12}$$

Hence the solution without consideration of fringing effect is obtained:

$$\Phi = \frac{V}{\varphi}\phi. \tag{13}$$

The intensity of electric field is computed as [5]

$$E = \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} = \frac{V}{\rho\varphi} \tag{14}$$

and the surface density of charge on the electrode plate is given by

$$\sigma = \varepsilon_0 E = \frac{\varepsilon_0 V}{\rho\varphi}. \tag{15}$$

Then the charge on the plate per unit of longitudinal length is

$$q = \int \sigma d\rho = \int_r^{r+l} \frac{\varepsilon_0 V}{\rho\varphi} d\rho = \frac{\varepsilon_0 V}{\varphi}\ln(1+\frac{l}{r}). \tag{16}$$

Therefore we obtain the inner capacitance per unit of longitudinal length

$$C_{in} = \frac{q}{V} = \frac{\varepsilon_0}{\varphi} \ln(1 + \frac{l}{r}). \tag{17}$$

4. The Fringing Capacitance

It is worth to stand out that the total capacitance obtained through conformal mapping is a general expression, including both the inner capacitance and the fringing capacitance raised from the field diffusion towards outside. Hence, the fringing capacitance per unit of longitudinal length should be the difference between the total one and the inner one:

$$C_{fr} = C - C_{in} = \varepsilon_0 \left[\frac{K'(k)}{K(k)} - \frac{1}{\varphi} \ln(1 + \frac{l}{r}) \right]. \tag{18}$$

To apparently see the significance of fringing capacitance in (18), let us compare (17) with (6). Note [6]

$$\frac{K'(k)}{K(k)} = -\frac{1}{\pi} \ln q \tag{19}$$

and introduce

$$\lambda = \frac{1 - \sqrt{k'}}{2(1 + \sqrt{k'})}. \tag{20}$$

From the computation of complete elliptic integration, we have [7]

$$\lambda = \frac{\vartheta_3 + \vartheta_4}{2(\vartheta_3 - \vartheta_4)} = \frac{q + q^5 + q^{25} + \cdots}{1 + 2q^4 + 2q^{16} + 2q^{35} + \cdots}, \tag{21}$$

where ϑ represents the ϑ-function. The solution of the above equation for q is [8]

$$q = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{35} + \cdots. \tag{22}$$

Equation (20) implies $\lambda < \frac{1}{2}$ owing to $0 < k' < 1$. Under the condition of

$$\lambda << 1, \tag{23}$$

we may only take the first term on the right side of (22) approximately. Thus (19) is rewritten as

$$\frac{K'(k)}{K(k)} = -\frac{1}{\pi}\ln\lambda. \tag{24}$$

Substituting (8), (3) and (2) into (20) yields, after simplification,

$$\lambda = \frac{1}{2}\left(\frac{l+r}{r}\right)^{\frac{\pi}{\varphi}}\left[1 - \sqrt{1 - \left(\frac{r}{l+r}\right)^{\frac{2\pi}{\varphi}}}\right]. \tag{25}$$

In the case of

$$\frac{r}{l+r} << 1, \tag{26}$$

we approximately use, neglecting the higher order terms,

$$\sqrt{1 - \left(\frac{r}{l+r}\right)^{\frac{2\pi}{\varphi}}} = 1 - \frac{1}{2}\left(\frac{r}{l+r}\right)^{\frac{2\pi}{\varphi}}. \tag{27}$$

Equation (25) then becomes

$$\lambda = \frac{1}{4}\left(\frac{r}{l+r}\right)^{\frac{\pi}{\varphi}}. \tag{28}$$

Therefore, from (24), we get

$$\frac{K'(k)}{K(k)} = \frac{1}{\varphi}\ln\left(1 + \frac{l}{r}\right) + \frac{2}{\pi}\ln 2. \tag{29}$$

Considering $0 < \varphi < \frac{\pi}{2}$ and $\frac{l}{r} >> 1,$ the second term on the right side of (29) is much less than the first one then could be neglected. So (29) becomes

$$\frac{K'(k)}{K(k)} = \frac{1}{\varphi}\ln\left(1 + \frac{l}{r}\right). \tag{30}$$

Substituting (30) into (6) leads to

$$C = \frac{\varepsilon_0}{\varphi}\ln\left(1 + \frac{l}{r}\right) \tag{31}$$

which is as the same as (17).

Consequently, (17) is an approximate expression for (6). The approximate conditions are r much less than l and φ less than $\pi / 2$ indicated in equations (25) and (29). The fringing effect in (18) can be neglected only as those restrictive conditions are ensured.

5. Conclusion

Computation of the fringing capacitance of an inclined plate capacitor is complicated due to the different sizes of two electrodes and their irregular locations. However, the conformal mapping provides an elegant way to calculate it accurately. Some skillful work has been done and the accurate result is expressed with the elliptic integral. To handle this problem by separating variables in solving Laplace equation must be restricted in following three approximate conditions: the two electrodes have the same size, sufficiently larger than their separation and the angle between the two plates is small. Only in such case the fringing effect can be neglected. The result achieved in this paper has general theoretical significance and is convenient for numerical computation on PC with math software such as MATHEMATICA, MAPPLE and MATLAB.

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