Fundamental J. Modern Physics, Vol. 2, Issue 1, 2011, Pages 15-22
Published online at http://www.frdint.com/

# BUILDING GAUGE THEORIES: THE NATURAL WAY

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**Abstract**

The construction of a gauge field theory for elementary particles usually starts by promoting global invariance of the matter action to a local one, this in turn implying the introduction of gauge fields. We present here a procedure that runs the other way: starting from the action for gauge fields, matter is gauge invariantly coupled on the basis of Lorentz invariance and charge conservation. This natural approach prevents using the concept of particles separated from gauge fields that mediate interactions.

Gauge invariance was introduced by H. Weyl [1] in 1919 in an attempt of unifying electrodynamics and general relativity. In his original proposal, gauging meant a space dependent change in the space-time unit of length together with the rescaling of the "compensating" or connection fields, the gauge potentials. Weyl's original proposal was incorrect but, nevertheless, the concept of gauge invariance survived as a central symmetry of Maxwell equations, extremely useful in the development of electrodynamics, particularly at the quantum level. At present, electromagnetic interactions of matter mediated by photons can be understood arising from the requirement of local Abelian gauge invariance in the theory. This idea was

Keywords and phrases: gauge theory building, particle physics.

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Communicated by Luis Anchordoqui
Received October 10, 2011

generalized by Yang and Mills [2] to the case of non-Abelian local symmetries in a work that can be considered as the starting point of the far-reaching analogy between electromagnetism and the other fundamental forces. The theoretical trail that was opened in 1954 has had a tremendous success as nowadays the fundamental interactions are described by local gauge theories [3].

Weyl's idea was presented long before its own time becoming today a fundamental ingredient of our present understanding of the fundamental interactions of Nature. It is important to stress that all these advances are based upon the basic concepts of gauge fields (the electromagnetic potentials and their non-Abelian generalizations) and the corresponding group of gauge transformations. The natural mathematical framework for this approach is given by the theory of fiber bundles [4].

Once one recognizes the connection between the electromagnetic field strength and the gauge field covariant derivative, it becomes manifest that gauge invariance is the paramount property of electrodynamics. Already in the non-relativistic quantum description of a charged particle in an electromagnetic field, gauge invariance imposes the introduction a phase factor in the particle wave function in order to maintain gauge covariance at the Hamiltonian level. As it is well known the phase factor is directly associated to the covariant derivative and in this way it becomes clear how gauge covariance is transmitted from the field strength to the particle wave function.

The standard way for constructing a gauge theory for matter fields seems to go in the opposite direction. Indeed, one starts by first choosing on physical grounds an appropriate unitary Lie group $G$ as the matter symmetry group and proposes an action governing the dynamics. The action should have a global invariance under $G$, i.e., the action should be invariant under constant phase transformations associated to the elements $g \in G$ and this implies the conservation of a Noether charge.

The second step in the usual construction of gauge invariant actions is to promote the global invariance to a local one, as in Yang-Mills proposal, so that $g \to d(x) \in \overline{G}$. Here we write $\overline{G}$ to indicate that now the group corresponds to local gauge transformations (i.e., the associated phases are space-time dependent). In order to have a gauge invariant matter field action, it is then mandatory to introduce gauge fields that compensate, via their gauge transformation, the changes produced by the local phase changes. As a result, the kinetic part of the matter Lagrangian turns to be written in terms of covariant derivatives. Now, once gauge fields are introduced, it becomes clear that a kinetic term for them should be included. Finally, under rather general assumptions for the behavior of $g(x)$ at infinity, the conserved Noether charge remains unchanged with respect to the global case.

It is the aim of this note to show that it is possible and natural to invert the process of theory building described above. Namely, the idea is to avoid as starting point the "decree" that promotes the originally global phase transformations of the matter Lagrangian to local ones. In our view, the proposal we present is closer to Weyl's ideas and their modern geometrical interpretation being tightly connected with the physical idea of interactions. Our main idea is to show that it is possible to build the theory of fundamental interactions starting from the gauge fields, the interaction mediators. Then, when sources are added, one obtains the corresponding matter-gauge field theory by only imposing the local gauge invariance and the Lorentz invariance of Maxwell equations, or their non-Abelian generalizations. To illustrate these ideas we begin with electrodynamics and afterwards we present a general non-Abelian case.

Let us start from the Maxwell equations. In the absence of sources, the "physical" pair of Maxwell equations read

$$\partial_\mu F^{\mu\nu} = 0 \tag{1}$$

with the field strength $F_{\mu\nu}(x)$ defined in terms of the $U(1)$ gauge field $A_\mu(x)$ as

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{2}$$

(The other pair is of course given by the Bianchi identity).

Local gauge transformations, related to the fact that $\vec{A}$ has a definite curl (the magnetic field $\vec{B}$). but its divergence is not fixed are the natural symmetry of Maxwell equations. Such transformations read

$$A_\mu(x) \to A_\mu^\Lambda(x) = A_\mu(x) + \partial_\mu \Lambda(x) \tag{3}$$

and leave the field strength $F_{\mu\nu}$ unchanged so that equation (1) is gauge invariant. Maxwell equations can be derived from the Lagrangian density

$$L_M = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \tag{4}$$

which is Lorentz invariant and gauge invariant.

In the presence of an external source $j^{\mu}_{\rm ext}$, equation (1) should be modified to

$$\partial_{\mu} F^{\mu\nu} = e j^{\nu}_{\rm ext} \tag{5}$$

with $e \in \mathbb{R}$, the coupling associated to the source. Within a Lagrangian formulation, the natural and simplest Lorentz invariant term coupling the source to the gauge field which leads to equation (5) is

$$L_{\rm int} = e A_{\mu}(x) j^{\mu}(x)_{\rm ext}. \tag{6}$$

Moreover, if gauge invariance of the Lagrangian

$$L = L_M + L_{\rm int} \tag{7}$$

is to be maintained, the external current should not change under gauge transformations

$$j^{\mu}_{ext}(x) \rightarrow j^{\mu\,\Lambda}_{ext}(x) = j^{\mu}_{ext}(x) \tag{8}$$

and satisfy

$$\partial_{\mu} j^{\mu}_{ext}(x) = 0. \tag{9}$$

Condition (8) is the natural one for an external (non-dynamical for the moment) current. Being the field strength antisymmetric, equation (9) is forced by Maxwell equations (5). All this ensures that the Lagrangian (6) changes at most as a total derivative under gauge transformations.

The Maxwell equations are a consequence of relativistic invariance, or better said, their structure is largely prescribed by Lorentz symmetry. In the same token one can anticipate the way matter should be coupled to gauge fields. Just to visualize our main idea for theory building let us now consider a dynamical Dirac fermion $\psi(x)$ at the origin of the four-vector current coupled to the gauge field. We call it $j^{\mu}(x)$ to distinguish it from the the previous external current and consider the (most economic) case of a bilinear combination of fermions. Lorentz invariance and gauge invariance guide the construction. Primo, Lorentz invariance forces to write the vector as

$$j^{\mu}(x) = \overline{\psi}(x)\gamma^{\mu}\psi(x), \tag{10}$$

where $\overline{\psi} = \psi^{\dagger}\gamma^{0}$. Secundo, gauge invariance of $j^{\mu}$ (a requirement analogous to (8)) imposes

$$\psi(x) \to \psi^{\Lambda}(x) = \exp(iq\Lambda(x))\psi \tag{11}$$

with $q \in \mathbb{R}$.

In order to make such fermions a dynamical field one should include a kinetic energy term. The simplest one is provided by the Dirac free Lagrangian

$$L_D = i\overline{\psi}(x)\gamma^{\mu}\partial_{\mu}\psi(x). \tag{12}$$

The total Lagrangian

$$L = L_M + L_D + L_{\text{int}} \tag{13}$$

will be gauge invariant provided one identifies $q$ in (11) with e in (5). Then, by means of the covariant derivative, $D_{\mu}$,

$$D_m = \partial_{\mu} - eA_{\mu}. \tag{14}$$

Lagrangian $L$ can be written in the usual form

$$L - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}\gamma^{\mu} D_{\mu}\psi \tag{15}$$

yielding the so-called "minimal electromagnetic coupling".

The extension to the non Abelian case is straightforward. We consider gauge fields $A_{\mu}(x) = A_{\mu}^{a}(x)t^{a}$ taking values in the Lie algebra of $SU(N)$ with generators $t^{a}$. Under a gauge transformation the gauge field changes as

$$A_{\mu}(x) \to A_{\mu}^{\Lambda}(x) = g^{-1}(x)A_{\mu}(x)g(x) - \frac{i}{e} g^{-1}(x)\partial_{\mu} g(x) \tag{16}$$

with

$$g(x) = \exp(i\Lambda(x)) \tag{17}$$

and the gauge phase is defined as $\Lambda(x) = \Lambda^{a}(x)t^{a}$. Infinitesimally one has

$$A_{\mu}(x) \to A_{\mu}^{\Lambda}(x) = A_{\mu} + \frac{1}{e} D_{\mu}[A]\Lambda(x) + O(\Lambda^{2}), \tag{18}$$

where $D_\mu[A]$ is the covariant derivative now defined as

$$D_\mu[A] = \partial_\mu + ie[A_\mu(x), ] \tag{19}$$

transforming according to

$$D_\mu[A^\Lambda] = g^{-1}(x) D_\mu[A] g(x). \tag{20}$$

The non-Abelian field strength $F_{\mu\nu} = F^a_{\mu\nu} t^a$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu . A_\nu] \tag{21}$$

and the Maxwell equations (1) are naturally extended to Yang-Mills equations

$$D_\mu[A] F^{\mu\nu} = 0 \tag{22}$$

Under gauge transformations the field strength changes covariantly,

$$F_{\mu\nu}(x) \to F^\Lambda_{\mu\nu}(x) = g^{-1}(x) F_{\mu\nu} g(x) \tag{23}$$

and so does equation (22). Of course, this equation can be derived from the Yang-Mills Lagrangian

$$L_{YM} = -\frac{1}{2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu}. \tag{24}$$

As for the Maxwell case, let us now introduce an external source $j^\mu_{\mathrm{ext}}$ in the Yang-Mills equations. One then has

$$D_\mu F^{\mu\nu} = e j^\nu_{\mathrm{ext}}, \tag{25}$$

where, for consistency, the current should take values in the Lie algebra of $SU(N)$

$$j^\mu(x)_{ext} = j^{\mu a}_{\mathrm{ext}}(x) t^a. \tag{26}$$

In order to derive equation (25) from a Lagrangian, the natural Lorentz invariant interaction term to add is

$$L_{\mathrm{int}} = e \mathrm{tr} A_\mu j^\mu_{\mathrm{ext}}. \tag{27}$$

Gauge invariance of the Lagrangian

$$L = L_{YM} + L_{\mathrm{int}} \tag{28}$$

will be guaranteed provided the external source changes according to

$$j^{\mu}_{\text{ext}}(x) \to j^{\mu\Lambda}_{\text{ext}}(x) = g^{-1}(x) j^{\mu}_{\text{ext}}(x) g(x) \tag{29}$$

and satisfies

$$D_{\mu}[A] j^{\mu}_{\text{ext}}(x) = 0. \tag{30}$$

Again, in order to make the source $j^{\mu}_{\text{ext}}$ in equation (25) dynamical, one should introduce Dirac fermions. Following the same lines as in the Abelian case, one defines a current $j^{\mu}$ taking values in the Lie algebra of $SU(N)$

$$j^{\mu a}(x) = \bar{\psi}^{i} \gamma^{\mu} T^{a}_{ij} \psi, \; i, \; j = 1, 2, ..., N, \tag{31}$$

where for simplicity we have taken fermions in the fundamental representation of $SU(N)$ with generators $T^{a}$ The transformation law

$$\psi(x) \Longrightarrow \psi^{\Lambda}(x) = \exp(iq\Lambda(x))\psi(x) \tag{32}$$

ensures that current (31) changes as in (29) provided $q = e$.

Adding the natural kinetic energy term for fermions one ends up with a total Lagrangian of the form

$$L = -\frac{1}{4} F^{a}_{\mu} F^{a\mu\nu} + \bar{\psi}_{i}(i\partial\delta_{ij} - A^{a} T^{a}_{ij})\psi_{j} \tag{33}$$

defining a gauge field theory of $SU(N)$.

One should note that in this non-Abelian case, the matter current $j_{\mu}$ is, according to equation (30), covariantly conserved and then it does not lead, by itself, to a conserved charge. It is only when the gauge Noether current $(j^{\text{gauge}})^{a}_{\mu} = f^{abc} F^{b}_{\mu\nu} A^{\nu c}$ associated to the Yang-Mills Lagrangian is included that a conserved charge $J_{\mu} = (j^{\text{gauge}})_{\mu} + j_{\mu}$ can be defined, verifying $\partial^{\mu} J_{\mu} = 0$.

As stated in the introduction, the usual way in which one builds up the Lagrangian of charged matter fields coupled to gauge fields is to promote the global unitary symmetry of the matter Lagrangian to a local (gauge) symmetry, this requiring the introduction of a gauge field. We have followed here the opposite way:

we started from the pure gauge field Lagrangian and considered the constraints imposed by Lorentz and gauge invariance when coupling a matter source ending with the same Lagrangian and conserved Noether current. Moreover, our presentation is parallel to the geometric approach to gauge theories where the starting point is to define a connection in a principal fiber bundle fiber and then introduce matter fields as sections in the associated vector bundle [5].

As we have shown, this procedure accepts generalizations to any gauge group. It does not take the matter field Lagrangian as a starting point but matter is gauge invariantly coupled on the basis of Lorentz invariance and charge conservation.

## Acknowledgements

Partially supported by ANPCyT, Argentina. We warmly thank illuminating comments by F. Cornet, F. del Aguila and J. Sánchez Guillén.

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